PERTURBATION FINITE VOLUME METHOD FOR CONVECTIVE-DIFFUSION INTEGRAL EQUATION*

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ABSTRACT: A perturbation finite volume (PFV) method for the convective-diffusion integral equation is developed in this paper. The PFV scheme is an upwind and mixed scheme using any higher-order interpolation and second-order integration approximations, with the least nodes similar to the standard three-point schemes, that is, the number of the nodes needed is equal to unity plus the face-number of the control volume. For instance, in the two-dimensional (2-D) case, only four nodes for the triangle grids and five nodes for the Cartesian grids are utilized, respectively. The PFV scheme is applied on a number of 1-D linear and nonlinear problems, 2-D and 3-D flow model equations. Comparing with other standard three-point schemes, the PFV scheme has much smaller numerical diffusion than the first-order upwind scheme (UDS). Its numerical accuracies are also higher than the second-order central scheme (CDS), the power-law scheme (PLS) and QUICK scheme.

KEY WORDS: perturbation finite volume, convective-diffusion integral equation, numerical accuracy

1 INTRODUCTION

The convective-diffusion equation is important in such fields as fluid mechanics and heat transfer. In a conventional finite volume treatment of the integral equation, as reviewed by Ferziger-Peric^[1], the discrete equation is obtained by two-level approximations, i.e. integral approximation and interpolation approximation. The diffusive flux integral is evaluated using the midpoint rule and the central difference scheme (CDS) interpolation of the normal derivative, which is the simplest and most widely used approximation. The convective flux integral is evaluated using the midpoint rule and either CDS or UDS interpolation. First-order upwind difference scheme (UDS) satisfies the boundedness criteria unconditionally, i.e. it will never yield oscillatory solutions^[1,2]. However, UDS is numerically diffusive, especially in three dimensional case, which will introduce a large diffusive error in both the streamwise and normal directions. The second-order CDS offers a good compromise between accuracy, simplicity of programming and code

maintenance, robustness and efficiency. However, the CDS will produce oscillatory solutions if grids are not fine enough. Higher-order schemes will also lead to oscillations on coarse grids but the results will converge more rapidly to an accurate solution as the grid is refined. As we know, any higher-order finite volume (FV) scheme requires a higher-order interpolation at more than one cell face location. This is manageable on a structured grid, but rather difficult on an unstructured grid. However, unstructured grids allow great flexibility in adapting the grid to the domain boundaries and flowfield. In general, control volumes (CV) of arbitrary shape, i.e. with any number of cell faces, may be used. Therefore, the FV scheme with formulation like UDS or CDS should be the best choice. Some researchers, such as Spalding^[3], Patankar^[4], Lilek-Peric^[5], proposed various lower and higher-order schemes to avoid non-physical oscillations and numerical diffusions. The hybrid scheme of Spalding^[3] switches from CDS to UDS at any node as the grid Peclet number is larger than 2. This is too restrictive and it reduces the accuracy^[1,4,5]. The

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power-law scheme (PLS) of Patankar^[4] is a blend of UDS and other two schemes and it provides a good approach to exponential scheme, with the actual accuracy higher than that of the hybrid scheme^[4~6]. The above four three-point schemes and another five-point scheme named QUICK scheme^[1,7] have been widely used in commercial codes. The QUICK scheme uses a quadratic upwind interpolation approximation, with a third-order truncation error on both uniform and non-uniform grids. However, the QUICK scheme has to use more nodes as compared with UDS, CDS, PLS and hybrid schemes. A perturbation finite volume (PFV) scheme was proposed by Gao^[8]. Its formulation is consistent with that of UDS, but its numerical diffusion is much smaller than that of UDS. In this paper, the PFV scheme is developed and used for the convective-diffusion integral equation.

2 DEVELOPMENT OF PERTURBATION FINITE VOLUME SCHEME FOR CONVECTIVE-DIFFUSION EQUATION

The perturbation finite volume (PFV) method^[8] uses a first-order upwind difference scheme (UDS) for the convective-diffusion integral equation as its starting point. The mass fluxes through each face of the control volume (CV) and the source term are modified and expanded into a power-series of the grid-spacing. The coefficients of the power-series are determined with the use of a space-splitting technique and the relations between the convective and diffusion fluxes. After some operations, the PFV scheme for the convective-diffusion integral equation is worked out.

The integral form of the scalar transport equation is

$$\int_S \rho \varphi \mathbf{u} \cdot \mathbf{n} ds = \int_S \Gamma \text{grad} \varphi \cdot \mathbf{n} ds + \int_{\Omega} q dv \quad (1)$$

where φ represents any transported variable, such as temperature, energy and components of flow velocity, ρ is the density, $\mathbf{u}$ is the velocity vector, Γ is the diffusion coefficient, and q is the source term. S and Ω are the surface and the volume or area of the control volume, respectively. A typical CV's face labeled "jf" (see Fig.1) is considered. Suppose that the line connecting the node P (CV center) and the node jp (adjacent CV center) is nearly orthogonal to the cell's j -face. The integral in Eq.(1) is evaluated by the midpoint rule. Then the first-order upwind difference scheme (UDS) for the integral Eq.(1) can be written as

Fig.1 A typical CV's face and adjacent nodes

$$\sum_{j=1}^J \left\{ \left[\frac{\Gamma}{|\mathbf{d}_j|^2} \mathbf{d}_j \cdot \mathbf{S}_j - \min(\dot{m}_{jf}, 0) \right] \varphi_{jp} - \left[\frac{\Gamma}{|\mathbf{d}_j|^2} \mathbf{d}_j \cdot \mathbf{S}_j + \max(\dot{m}_{jf}, 0) \right] \varphi_p \right\} + q_p \Omega = 0 \quad (2)$$

The continuity equation gives

$$\sum_{j=1}^J \dot{m}_{jf} = 0 \quad (3)$$

where $\dot{m}_{jf}$ is the mass flux through the j -face, $j = 1, 2, \dots, J$. For a 2-D problem, $J = 3$ and 4 indicate the triangle and Cartesian grids, respectively. φ_p is the value of φ at the center of the control volume (CV), φ_{jp} is the value of φ at the adjacent CV's center. $\mathbf{S}_j$ is the area vector of the j -face and its direction agrees with the outer normal of the j -face. $\mathbf{d}_j$ is the vector linking two center nodes p and jp with the direction from p to jp . According to above assumptions, the direction of $\mathbf{d}_j$ is nearly consistent with that of $\mathbf{S}_j$ and Ω is the volume of the control volume.

Equation (2) is split into

$$\left[\frac{\Gamma}{|\mathbf{d}_1|^2} \mathbf{d}_1 \cdot \mathbf{S}_1 - \min(\dot{m}_{1f}, 0) \right] \varphi_{1p} - \left[\frac{\Gamma}{|\mathbf{d}_1|^2} \mathbf{d}_1 \cdot \mathbf{S}_1 + \max(\dot{m}_{1f}, 0) \right] \varphi_p + F_1 = 0 \quad (4.1)$$

⋮

$$\left[\frac{\Gamma}{|\mathbf{d}_j|^2} \mathbf{d}_j \cdot \mathbf{S}_j - \min(\dot{m}_{jf}, 0) \right] \varphi_{jp} - \left[\frac{\Gamma}{|\mathbf{d}_j|^2} \mathbf{d}_j \cdot \mathbf{S}_j + \max(\dot{m}_{jf}, 0) \right] \varphi_p + q_j \Omega + F_j = 0 \quad (4.j)$$

⋮

$$\left[\frac{\Gamma}{|\mathbf{d}_J|^2} \mathbf{d}_J \cdot \mathbf{S}_J - \min(\dot{m}_{Jf}, 0) \right] \varphi_{Jp} - \left[\frac{\Gamma}{|\mathbf{d}_J|^2} \mathbf{d}_J \cdot \mathbf{S}_J + \max(\dot{m}_{Jf}, 0) \right] \varphi_p + q_J \Omega + F_J = 0$$

$$\max(\dot{m}_{jf}, 0) \varphi_p + F_j = 0 \quad (4.J)$$

where F_j expresses the contribution of other directions to the j -face flux. Then we have

$$\sum_{j=1}^J F_j = 0 \quad (5)$$

In the PFV method^[8], in order to improve UDS's accuracy, both the mass fluxes $\dot{m}_{jf}$ and the source term q_p are modified. Suppose that there are perturbations on the values of the mass fluxes $\dot{m}_{jf}$ and the source term q_p ,

$$\dot{m}_{jfp} = \dot{m}_{jf} + \sum_{n=1} A_{jn} |\mathbf{d}_j|^n \quad (6.1)$$

$$q_{pp} = q_p + \sum_{n=1} B_{jn} |\mathbf{d}_j|^n \quad (6.2)$$

and F_j also needs to be expanded into a power-series of $|\mathbf{d}_j|$ i.e.

$$F_{jp} = F_j + \sum_{n=1} C_{jn} |\mathbf{d}_j|^n \quad (6.3)$$

After φ_{jp} and φ_p in Eq.(4.j) are expanded into Taylor series, we have

$$\begin{aligned} & \left[\frac{\Gamma \mathbf{d}_j \cdot \mathbf{S}_j}{|\mathbf{d}_j|^2} - \min \left(\left(\dot{m}_{jf} + \sum_{n=1} A_{jn} |\mathbf{d}_j|^n \right), 0 \right) \right] \cdot \\ & \left[\varphi_{jf} + \frac{\partial \varphi_{jf}}{\partial \xi_j} \delta_j |\mathbf{d}_j| + \frac{1}{2!} \frac{\partial^2 \varphi_{jf}}{\partial \xi_j^2} \delta_j^2 |\mathbf{d}_j|^2 + \right. \\ & \left. \frac{1}{3!} \frac{\partial^3 \varphi_{jf}}{\partial \xi_j^3} \delta_j^3 |\mathbf{d}_j|^3 + \dots \right] - \left[\frac{\Gamma \mathbf{d}_j \cdot \mathbf{S}_j}{|\mathbf{d}_j|^2} + \right. \\ & \left. \max \left(\left(\dot{m}_{jf} + \sum_{n=1} A_{jn} |\mathbf{d}_j|^n \right), 0 \right) \right] \cdot \\ & \left[\varphi_{jf} - \frac{\partial \varphi_{jf}}{\partial \xi_j} (1 - \delta_j) |\mathbf{d}_j| + \frac{1}{2!} \frac{\partial^2 \varphi_{jf}}{\partial \xi_j^2} (1 - \delta_j)^2 |\mathbf{d}_j|^2 - \right. \\ & \left. \frac{1}{3!} \frac{\partial^3 \varphi_{jf}}{\partial \xi_j^3} (1 - \delta_j)^3 |\mathbf{d}_j|^3 + \dots \right] + \\ & \left[q_p + \sum_{n=1} B_{jn} |\mathbf{d}_j|^n \right] \Omega + \left[F_j + \sum_{n=1} C_{jn} |\mathbf{d}_j|^n \right] = 0 \quad (7) \end{aligned}$$

where $\frac{\partial \varphi}{\partial \xi_j}$ is the derivative of φ with respect to the direction of ξ_j . $\delta_j |\mathbf{d}_j|$ is the distance from the j -face to node jp . In the discrete approximation, all of $\Gamma \mathbf{d}_j \cdot \mathbf{S}_j / |\mathbf{d}_j|^2$, $\dot{m}_{jf}$, q_p and F_j are regarded as constants, then we can obtain the following relations from

Eq.(4.j)

$$\frac{\partial^n \varphi_{jf}}{\partial \xi_j^n} = \dot{m}_{jf}^{n-1} \frac{|\mathbf{d}_j|^n}{(\Gamma \mathbf{d}_j \cdot \mathbf{S}_j)^n} (\dot{m}_{jf} \varphi_{jf} - q_p \Omega - F_j) \quad (8)$$

$$n = 1, 2, 3, \dots$$

Substituting relations (8) into Eq.(7), comparing the coefficients of $|\mathbf{d}_j|^n$ ($n = 1, 2, \dots$), we can obtain

$$A_{jn} |\mathbf{d}_j|^n = \frac{1}{(n+1)!} \dot{m}_{jf} R_{jf}^n (\text{sign} \dot{m}_{jf})^n \quad (9.1)$$

$$(B_{jn} + C_{jn}) |\mathbf{d}_j|^n = \frac{1}{(n+1)!} (q_p \Omega + F_j) R_{jf}^n (\text{sign} \dot{m}_{jf})^n \quad (9.2)$$

$$R_{jf} = \dot{m}_{jf} |\mathbf{d}_j| / \Gamma S_j = \rho U_{\xi_j} |\mathbf{d}_j| / \Gamma \quad (9.3)$$

where U_{ξ_j} is the velocity component in the ξ_j direction, $|\mathbf{d}_j|$ the distance from p to jp . R_{jf} can be regarded as the grid Reynolds number or the grid Peclet number in the ξ_j direction. It should be noted that coefficients A_{jn} , B_{jn} and C_{jn} do not include the distance-factor δ_j . It means that the PFV scheme does not make any difference to the fact how different are the distances from the j -face center jj to the nodes of p and jp .

A similar treatment is made for other equations of (4). Substituting All A_{jn} , B_{jn} , $\dots$ into Eqs.(4), summing them, we obtain the following perturbation finite volume (PFV) scheme

$$\begin{aligned} & \sum_{j=1}^J \frac{1}{G_j} \left\{ \left[\frac{\Gamma \mathbf{d}_j \cdot \mathbf{S}_j}{|\mathbf{d}_j|^2} - \min(\dot{m}_{jf} G_j, 0) \right] \varphi_{jp} - \right. \\ & \left. \left[\frac{\Gamma \mathbf{d}_j \cdot \mathbf{S}_j}{|\mathbf{d}_j|^2} + \max(\dot{m}_{jf} G_j, 0) \right] \varphi_p \right\} + q_p \Omega = 0 \quad (10) \end{aligned}$$

where

$$G_j = \sum_{n=0}^N \frac{1}{(n+1)!} R_{jf}^n (\text{sign} \dot{m}_{jf})^n \quad (10.1)$$

$$\text{sign} \dot{m}_{jf} = \begin{cases} 1 & m_{jf} > 0 \\ -1 & m_{jf} < 0 \end{cases} \quad (10.2)$$

3 SEVERAL PROPERTIES OF PERTURBATION FINITE VOLUME SCHEME

If let $N = 0$ in relation (10.1), the PFV scheme is returned to the original UDS of Eq.(2). From the treatment of deriving the PFV scheme, we know that the truncation-error of the modified differential equation is of $(N+1)$ th-order. Therefore, the PFV scheme (10) is a mixed scheme with $(N+1)$ th-order interpolation and second-order integration approximations.

The theoretical accuracy of the scheme (10) is still of second-order. However, as pointed out by some researchers such as Ferziger-Peric^[1] and Lilek-Peric^[5], the actual numerical accuracy of a higher-order interpolation approximation is higher than those of UDS or CDS. This conclusion is again verified by the present computational results given in the next section.

A meaningful fact is that the distance factor δ_j does not appear in the formulation of the PFV scheme (10). Hence the PFV scheme with $\delta_j \neq 1/2$ is the same as that with $\delta_j = 1/2$. This is a good property for the present PFV scheme. It is well known that δ_j is often not equal to $1/2$ for practical unstructured and structured grids. On the other hand, for the Cartesian grid we may let δ_j tend to unity, i.e., the j -face center tends to the node p , see Fig.2. Then the PFV scheme can be transformed into the following form

Fig.2 Cartesian grid

$$\begin{aligned} & \frac{2\Gamma\Delta y}{\Delta x G_{px}} \{ [1 - \min(G_{px}R_{\Delta x}, 0)]\varphi_{i+1,j} - \\ & [2 + \max(G_{px}R_{\Delta x}, 0) - \min(G_{px}R_{\Delta x}, 0)]\varphi_{i,j} + \\ & [1 + \max(G_{px}R_{\Delta x}, 0)]\varphi_{i-1,j} \} + \\ & \frac{2\Gamma\Delta x}{\Delta y G_{py}} \{ [1 - \min(G_{py}R_{\Delta y}, 0)]\varphi_{i,j+1} - \\ & [2 + \max(G_{py}R_{\Delta y}, 0) - \min(G_{py}R_{\Delta y}, 0)]\varphi_{i,j} + \\ & [1 + \max(G_{py}R_{\Delta y}, 0)]\varphi_{i,j-1} \} + q_p\Omega = 0 \quad (11) \end{aligned}$$

where $R_{\Delta x} = \rho u_{i,j} \Delta x / \Gamma$, $R_{\Delta y} = \rho u_{i,j} \Delta y / \Gamma$

$$G_{px} = R_{\Delta x} \sum_{n=0}^N \frac{1}{(n+1)!} R_{\Delta x}^n (\text{sign} R_{\Delta x})^n \quad (11.1)$$

$$G_{py} = R_{\Delta y} \sum_{n=0}^N \frac{1}{(n+1)!} R_{\Delta y}^n (\text{sign} R_{\Delta y})^n \quad (11.2)$$

u_{ij} and v_{ij} are the velocity components at the node $p(i, j)$ in the x - and y -directions, respectively. The scheme (11) is just the perturbation finite difference (PFD) scheme for the convective-diffusion equation developed by Gao^[9]. If we let $N \rightarrow \infty$ in relations (11.1) and (11.2), the PFD scheme (11) can be transformed to an exact numerical scheme for the convective-diffusion equation, as shown in Appendix A. The above facts show the advantages of a higher-order interpolation approximation in the finite volume method.

4 COMPUTATIONAL RESULTS

The PFV scheme developed in the previous sections are applied on a number of linear and nonlinear problems, which include boundary-layer problems, elliptic singular perturbation problems and two- and three-dimensional Navier-Stokes equations. For linear one-dimensional problems, the Gauss-Seidel iteration with SLOR procedure is used and for nonlinear and two-, three-dimensional problems, a tri-diagonal system also with SLOR procedure is employed.

The iterative procedure is repeated until $|u^{(n+1)} - u^{(n)}| < \delta$ for all grids, where n is the iteration number and $\delta = 10^{-10}$ is the selected tolerance. The whole calculations are carried out with double precision. The results are compared with the corresponding exact solution or the numerical solutions of the standard three-point schemes, which include first-order upwind scheme (UDS), second-order central scheme (CDS) and the power-law scheme (PLS). In addition, the five-point QUICK scheme is also used for the linear one-dimensional problems. The second-order to twelfth-order interpolation PFV schemes are utilized for all calculations and the sixth-order interpolation PFV scheme is used for the comparison with other standard schemes.

4.1 One-dimensional Constant Coefficient Problems

4.1.1 Problem 1

$$\begin{aligned} & -\varepsilon u_{xx} + u_x = 0 \\ & u(0) = 0 \quad u(1) = 1 \end{aligned} \quad (12)$$

The exact solution is given by $u(x) = (e^{x/\varepsilon} - 1)/(e^{1/\varepsilon} - 1)$.

As shown in Fig.3, for the small value of ε , the problem is a boundary-layer type problem. On the most region, the solution approximates the upstream

value $u(0)$, only in a very thin-layer near the downstream boundary, it increases rapidly to the downstream values $u(1)$. The calculations are carried out for $\varepsilon = 10^{-2}$, 5×10^{-3} , 10^{-3} and the computational region $0 < x < 1$ is distributed uniformly with 1001 grid points ($h = 1/1000$). The maximum absolute errors of the PFV scheme with the second- to twelfth-order interpolation approximations (corresponding 2PFV to 12PFV) are given in Table 1. Only for 10^{-3} , the maximum error continues to decrease with the order of the PFV interpolation approximation. For $\varepsilon = 10^{-2}$ and 5×10^{-3} the maximum error is nearly unchanged for the scheme with the order of interpolation approximation greater than six, thus the sixth-order PFV interpolation scheme is used for the comparison of accuracy. The comparison of the PFV scheme with QUICK and other normal three-point schemes is shown in Table 2. As expected, the first upwind scheme (UDS) produces the maximum absolute

error as compared to the second central scheme (CS), QUICK scheme and the power-law scheme (PLS), and our PFV scheme gives a higher accuracy for the problem.

Fig.3 The exact solution of problem 1

Table 1 Maximum absolute error for second- to twelfth-order interpolation upper PFV schemes (Problem 1)

ε	2PFV	4PFV	6PFV	8PFV	10PFV	12PFV
10^{-2}	5.6939×10^{-4}	2.5285×10^{-7}	3.5448×10^{-8}	3.5497×10^{-8}	3.5497×10^{-8}	3.5497×10^{-8}
5×10^{-3}	2.1198×10^{-3}	4.1465×10^{-6}	6.5423×10^{-9}	8.8682×10^{-9}	8.8699×10^{-9}	8.8699×10^{-9}
10^{-3}	3.2121×10^{-2}	1.3513×10^{-3}	3.0625×10^{-5}	4.1374×10^{-7}	3.4920×10^{-9}	2.7515×10^{-10}

Table 2 Maximum absolute error for PFV, QUICK and other standard three-point schemes (Problem 1)

ε	6PFV	UDS	CDS	PLS	QUICK
10^{-2}	3.5448×10^{-8}	1.7664×10^{-2}	3.0694×10^{-4}	5.7718×10^{-5}	1.3100×10^{-4}
5×10^{-3}	6.5423×10^{-9}	3.3998×10^{-2}	1.2316×10^{-3}	2.1758×10^{-4}	4.4107×10^{-4}
10^{-3}	3.0625×10^{-5}	1.3212×10^{-1}	3.4546×10^{-2}	3.3835×10^{-3}	1.1290×10^{-3}

4.1.2 Problem 2

$$\begin{aligned}
 -\varepsilon u_{xx} + u_x &= \varepsilon \pi^2 \sin \pi x + \pi \cos \pi x \\
 u(0) &= 0 \quad u(1) = 1
 \end{aligned}
 \tag{13}$$

The exact solution is given by $u(x) = \sin \pi x + (e^{x/\varepsilon} - 1)/(e^{1/\varepsilon} - 1)$ and is shown in Fig.4.

The same conditions with $h = 1/1000$ and $\varepsilon = 10^{-2}$, 5×10^{-3} , 10^{-3} are taken for the calculation. The comparison of the results is given in Table 3 and Table 4. The maximum error is nearly unchanged with the order of interpolation approximation greater than four for $\varepsilon = 10^{-2}$ and 5×10^{-3} and the order of interpolation approximation greater than six for $\varepsilon = 10^{-3}$. The PFV scheme also gives a higher accuracy than those of QUICK scheme and other three-

point schemes.

Fig.4 The exact solution of problem 2

Table 3 Maximum absolute errors for second- to twelfth-order interpolation upper PFV schemes (Problem 2)

ε	2PFV	4PFV	6PFV	8PFV	10PFV	12PFV
10^{-2}	4.77706×10^{-4}	5.16723×10^{-5}	5.163972×10^{-5}	5.163926×10^{-5}	5.163926×10^{-5}	5.163926×10^{-5}
5×10^{-3}	1.93994×10^{-3}	1.04446×10^{-4}	1.041566×10^{-4}	1.041563×10^{-4}	1.041563×10^{-4}	1.041563×10^{-4}
10^{-3}	3.14923×10^{-2}	1.01304×10^{-3}	5.15353×10^{-4}	5.14907×10^{-4}	5.14901×10^{-4}	5.14901×10^{-4}

Table 4 Maximum absolute errors for the PFV, QUICK and other standard three-point schemes (Problem 2)

ε	6PFV	UDS	CDS	PLS	QUICK
10^{-2}	5.1639×10^{-5}	1.5735×10^{-2}	3.0689×10^{-4}	6.0650×10^{-5}	1.3105×10^{-4}
5×10^{-3}	1.0416×10^{-4}	3.2119×10^{-2}	1.2316×10^{-3}	1.4484×10^{-4}	4.4111×10^{-4}
10^{-3}	5.1535×10^{-4}	1.3055×10^{-1}	3.4546×10^{-2}	3.0261×10^{-3}	1.1290×10^{-3}

4.1.3 Problem 3

$$\begin{aligned} \varepsilon u_{xx} + u_x - \varepsilon u &= -\sin x \quad 0 < x < \pi \\ u(0) &= 0 \quad u(\pi) = 0 \end{aligned} \quad (14)$$

The exact solution of the problem depicted in Fig.5 is given by

$$u(x) = \frac{1}{1+4\varepsilon^2} (Ae^{r_1 x/\pi} + Be^{r_2 x/\pi} + 2\varepsilon \sin x + \cos x)$$

where

$$r_1 = -\pi(1 + \sqrt{1+4\varepsilon^2})/2\varepsilon$$

$$r_2 = 2\pi\varepsilon(1 + \sqrt{1+4\varepsilon^2})^{-1}$$

$$A = (1 + e^{r_2})/(e^{r_1} - e^{r_2})$$

$$B = (1 + e^{r_1})/(e^{r_2} - e^{r_1})$$

The problem has a boundary layer at $x = 0$. The calculations are carried out with the grid interval

number of $n = 1000$. Maximum absolute errors obtained are given in Tables 5 and 6. Maximum errors are nearly unchanged for the order of PFV scheme greater than six and the new PFV scheme gives a higher accuracy.

Fig.5 The exact solution of problem 3

Table 5 Maximum absolute errors for second- to twelfth-order interpolation upper PFV schemes (Problem 3)

ε	2PFV	4PFV	6PFV	8PFV	10PFV	12PFV
10^{-2}	9.49495×10^{-3}	8.10751×10^{-5}	8.04810×10^{-5}	8.04796×10^{-5}	8.04796×10^{-5}	8.04796×10^{-5}
5×10^{-3}	3.05634×10^{-2}	5.49990×10^{-4}	1.61181×10^{-4}	1.61778×10^{-4}	1.61778×10^{-4}	1.61778×10^{-4}

Table 6 Maximum absolute errors for PFV, QUICK and standard three-point schemes (Problem 3)

ε	6PFV	UDS	CDS	PLS	QUICK
10^{-2}	8.0484×10^{-5}	1.3204×10^{-1}	6.0125×10^{-3}	1.4385×10^{-3}	3.7667×10^{-3}
5×10^{-3}	1.6188×10^{-4}	3.2109×10^{-1}	2.4298×10^{-2}	6.8345×10^{-3}	2.3023×10^{-2}

4.2 One-dimensional Variable Coefficient

Problems

4.2.1 Problem 4

$$\varepsilon u_{xx} - \frac{1}{1+x} u_x - \frac{1}{2+x} u = \left(\varepsilon - \frac{1}{1+x} - \right.$$

$$\frac{1}{2+x})e^x - \frac{1}{2+x}2^{-1/\varepsilon}(1+x)^{1+1/\varepsilon} \quad (15)$$

where $0 < x < 1$, $u(0) = 1 + 2^{-1/\varepsilon}$, $u(1) = e + 2$. The exact solution of the problem is given by $u(x) = e^x + 2^{-1/\varepsilon}(1+x)^{1+1/\varepsilon}$ and shown in Fig.6.

The calculations are also carried out for the value of $\varepsilon = 10^{-2}$, 5×10^{-3} , 10^{-3} with the grid interval number $n = 1000$. Maximum absolute errors obtained are given in Tables 7 and 8. The same conclusion can be obtained, that is, the maximum errors are unchanged with the order of interpolation scheme greater than four and the new PFV scheme gives a higher accuracy compared with other standard schemes.

Fig.6 The exact solution of problem 4

Table 7 Maximum absolute errors for second- to twelfth-order interpolation upper PFV schemes (Problem 4)

ε	2PFV	4PFV	6PFV	8PFV	10PFV	12PFV
10^{-2}	3.06550×10^{-4}	6.02289×10^{-6}	6.01760×10^{-6}	6.01760×10^{-6}	6.01760×10^{-6}	6.01760×10^{-6}
5×10^{-3}	1.16033×10^{-3}	1.30536×10^{-5}	1.30090×10^{-5}	1.30090×10^{-5}	1.30090×10^{-5}	1.30090×10^{-5}
10^{-3}	2.17132×10^{-2}	2.84073×10^{-4}	6.89833×10^{-5}	6.89532×10^{-5}	6.89530×10^{-5}	6.89530×10^{-5}

Table 8 Maximum absolute error for PFV, QUICK and standard three-point schemes (Problem 4)

ε	6PFV	UDS	CDS	PLS	QUICK
10^{-2}	6.0176×10^{-6}	1.8177×10^{-2}	1.5521×10^{-4}	3.3388×10^{-5}	6.7203×10^{-5}
5×10^{-3}	1.3009×10^{-5}	3.5477×10^{-2}	6.1716×10^{-4}	1.2280×10^{-4}	2.5476×10^{-4}
10^{-3}	6.8983×10^{-5}	1.5326×10^{-1}	1.5776×10^{-2}	2.3093×10^{-3}	2.7961×10^{-3}

4.2.2 Problem 5

$$u \frac{\partial u}{\partial x} = \frac{1}{Re} \frac{\partial^2 u}{\partial x^2} \quad 0 < x < 1 \quad (16)$$

with boundary conditions satisfied by the solution

$$u(x) = \tanh[Re(1-2x)/4]$$

The Burgers equation is regarded as a model equation of fluid flow. For large values of the Reynolds number Re , the solution contains an abrupt change at the point $x = 0.5$. To resolve the steep changes and also for more accurate representation, the given interval $[0, 1]$ is subdivided into three intervals $[0, 0.5 - 20\varepsilon]$, $[0.5 - 20\varepsilon, 0.5 + 20\varepsilon]$, $[0.5 + 20\varepsilon, 1]$ where $\varepsilon = 1/Re$. The number of subdivisions for the corresponding subintervals is selected as 10, 100 and 10, respectively. The Reynolds number is taken as $Re = 1000$. For a comparison purpose, the case

when the computational region $0 \leq x \leq 1$ is divided into 120 intervals uniformly is also calculated. The maximum absolute errors calculated are shown in Tables 9 and 10. For the order of PFV scheme greater than six (non-uniform grid) or greater than four (uniform grid), the maximum errors are unchanged. For the uniform grid with the interval number of 120, the solutions near the central point 0.5 are presented in Fig.7. The PFV scheme can be used for the non-uniform grid directly. For the non-uniform grid, the PFV scheme gives the same accuracy as the second-order scheme and the power-law scheme and it is higher than the first-upwind scheme. For the uniform grid, in the near shock region, the second-order central scheme produces non-physical oscillation and the first-order upwind scheme produces a larger dissipation, the maximum errors for these two schemes are larger than those of the PFV and PLS schemes.

Table 9 Maximum absolute errors for second- to twelfth-order interpolation upper PFV schemes (Problem 5)

Grid	2PFV	4PFV	6PFV	8PFV	10PFV	12PFV
$10 \times 100 \times 10$	4.29701×10^{-3}	2.52311×10^{-3}	2.52180×10^{-3}	2.52180×10^{-3}	2.52180×10^{-3}	2.52180×10^{-3}
120	3.86978×10^{-2}	1.31194×10^{-2}	8.76356×10^{-3}	7.78771×10^{-3}	7.59844×10^{-3}	7.57118×10^{-3}

Table 10 Maximum absolute errors for PFV, QUICK and three-point schemes (Problem 5)

Grid	6PFV	UDS	CDS	PLS
$10 \times 100 \times 10$	2.5218×10^{-3}	3.4357×10^{-2}	1.9563×10^{-3}	2.6277×10^{-3}
120	8.7636×10^{-3}	1.1123×10^{-1}	1.9491×10^{-1}	7.8333×10^{-3}

Fig.7 Solution near the centre point for $Re = 1000$ with uniform grid

4.3 Two-dimensional Case

4.3.1 Problem 6

Consider the differential equation

$$u_{xx} + u_{yy} + \omega u_x = 0 \quad 0 < x < 1 \quad (17)$$

with the boundary conditions satisfied by the solution

$$u(x, y) = \left(y(1 - y) - \frac{2x}{\omega} \right) e^{-\omega x}.$$

The problem has a boundary layer. The uniform grid with 40×40 intervals in the x - and y - directions is used for the computation for $\omega = 40$. The tri-diagonal matrix method with SLOR is used to obtain the solution. The results are presented in Table 11. The new three-point PFV scheme enjoys a higher accuracy.

Table 11 Maximum absolute errors for second- to twelfth-order interpolation upper PFV schemes (Problem 6)

Grid	2PFV	4PFV	6PFV	8PFV	10PFV	12PFV
40×40	3.28650×10^{-2}	1.77123×10^{-3}	4.37974×10^{-5}	6.11616×10^{-7}	9.36463×10^{-9}	1.31757×10^{-8}

Table 12 Maximum absolute error for PFV and other standard three-point schemes (Problem 6)

Grid	6PFV	UDS	CDS	PLS
40×40	4.3797×10^{-5}	9.1510×10^{-2}	8.5934×10^{-3}	2.7858×10^{-3}

4.3.2 Problem 7

Consider the following two-dimensional model-equations for fluid flow

$$u_{xx} + u_{yy} - uu_x - vv_y = (2 \sin y + \sin x) \cos x$$

$$v_{xx} + v_{yy} - uv_x - vv_y = (\sin y - 2 \sin x) \cos y \quad (18)$$

$$0 < x, y < \pi$$

which have the exact solution

$$\begin{aligned} u(x, y) &= -\cos x \sin y \\ v(x, y) &= \sin x \cos y \end{aligned} \quad (19)$$

The terms on the right-hand side of (18) do not themselves constitute possible pressure gradients to be truly representative of the two-dimensional Navier-

Stokes equations, but (19) satisfies both the equation of continuity and the differential equations (18). Thus the essential features of a model solution of the Navier-Stokes equations are preserved.

The uniform grid with 20×20 intervals in the x - and y - directions is used for the computation. The tri-diagonal matrix solutions with SLOR approach are treated alternatively for u and v equations. The initial values of u and v in the inner region are taken as the averaged values of four corner points. For the convergence, a tolerance $\delta = 10^{-10}$ is assumed. The iterative procedure converges rapidly. The maximum absolute errors for the u and v values are shown in Tables 13 and 14. The results for $u(0.7\pi, y)$ and $v(0.7\pi, y)$ with the values of y/π from 0.1 to 0.5 are given in Table 15 and Table 16.

Table 13 Maximum absolute errors for second- to twelfth-order interpolation upper PFV schemes (Problem 7)

	2PFV	4PFV	6PFV	8PFV	10PFV	12PFV
u	1.70889×10^{-3}	1.67845×10^{-3}	1.67832×10^{-3}	1.67832×10^{-3}	1.67832×10^{-3}	1.67832×10^{-3}
v	6.85069×10^{-4}	5.36321×10^{-4}	5.36283×10^{-4}	5.36283×10^{-4}	5.36283×10^{-4}	5.36283×10^{-4}

Table 14 Maximum absolute errors for PFV and other standard three-point schemes (Problem 7)

	6PFV	UDS	CDS	PLS
u	$1.678\,3\times 10^{-3}$	$4.807\,5\times 10^{-3}$	$1.686\,4\times 10^{-3}$	$1.681\,6\times 10^{-3}$
v	$5.362\,8\times 10^{-4}$	$8.095\,1\times 10^{-3}$	$4.837\,7\times 10^{-4}$	$5.483\,8\times 10^{-4}$

Table 15 Solutions of problem 7 with $h = \pi/20$ for $u(0.7\pi, y)$

y/π	Exact	6PFV	UDS	CDS	PLS
0.1	0.181 635 65	0.182 224 73	0.180 405 42	0.182 267 34	0.182 216 93
0.2	0.345 491 54	0.346 489 03	0.343 500 21	0.346 543 84	0.346 478 89
0.3	0.475 528 31	0.476 817 72	0.473 793 86	0.476 841 11	0.476 813 17
0.4	0.559 017 05	0.560 492 92	0.558 354 31	0.560 467 38	0.560 497 26
0.5	0.587 785 30	0.589 326 65	0.587 894 78	0.589 277 83	0.589 335 26

Table 16 Solutions of problem 7 with $h = \pi/20$ for $v(0.7\pi, y)$

y/π	Exact	6PFV	UDS	CDS	PLS
0.1	0.769 420 85	0.769 204 65	0.763 332 51	0.769 280 52	0.769 190 19
0.2	0.654 508 46	0.654 055 20	0.646 453 46	0.654 133 73	0.654 039 96
0.3	0.475 528 22	0.475 006 12	0.469 552 19	0.475 048 76	0.474 997 74
0.4	0.249 999 96	0.249 652 42	0.247 918 43	0.249 663 84	0.249 650 23
0.5	-0.000 000 04	-0.000 000 01	-0.000 000 01	-0.000 000 01	-0.000 000 01

4.4 Three-dimensional Case

Problem 8

Similar to the above two-dimensional case, we design a three-dimensional model equation of fluid flow as

$$\begin{aligned}
 &uu_x + vu_y + wu_z = u_{xx} + \\
 &\quad u_{yy} + u_{zz} + f_1(x, y, z) \\
 &uv_x + vv_y + wv_z = v_{xx} + \\
 &\quad v_{yy} + v_{zz} + f_2(x, y, z) \\
 &uw_x + vw_y + ww_z = w_{xx} + \\
 &\quad w_{yy} + w_{zz} + f_3(x, y, z)
 \end{aligned}
 \quad 0 < x, y, z < \pi \quad (20)$$

where $f_1(x, y, z)$, $f_2(x, y, z)$ and $f_3(x, y, z)$ are determined by the exact solution

$$\begin{aligned}
 u(x, y, z) &= -\cos x(\sin y + \sin z) \\
 v(x, y, z) &= \cos y(\sin x + \sin z) \\
 w(x, y, z) &= -\cos z(\sin y - \sin x)
 \end{aligned}
 \quad (21)$$

The numerical solution is obtained with the uniform grid sizes $h_1 = h_2 = h_3 = \pi/20$. The tri-diagonal matrix solutions with SLOR approach are treated alternatively for u , v and w equations. The factor of relaxation is set as 0.3. The initial values of u , v and w in the inner region are taken as the averaged values of eight corner points. For the convergence, a tolerance $\delta = 10^{-10}$ is assumed. The iterative procedure converges rapidly. The maximum absolute errors for the u , v and w values are shown in Tables 17 and 18.

Table 17 Maximum absolute errors for second- to twelfth-order interpolation upper PFV schemes (Problem 8)

	2PFV	4PFV	6PFV	8PFV	10PFV	12PFV
u	$5.972\,25\times 10^{-3}$	$5.464\,22\times 10^{-3}$	$5.460\,48\times 10^{-3}$	$5.460\,47\times 10^{-3}$	$5.460\,47\times 10^{-3}$	$5.460\,47\times 10^{-3}$
v	$4.626\,66\times 10^{-3}$	$3.890\,07\times 10^{-3}$	$3.891\,21\times 10^{-3}$	$3.891\,22\times 10^{-3}$	$3.891\,22\times 10^{-3}$	$3.891\,22\times 10^{-3}$
w	$1.705\,66\times 10^{-3}$	$1.603\,48\times 10^{-3}$	$1.603\,04\times 10^{-3}$	$1.603\,04\times 10^{-3}$	$1.603\,04\times 10^{-3}$	$1.603\,04\times 10^{-3}$

Table 18 Maximum absolute errors for PFV and other standard three-point schemes (Problem 8)

	6PFV	UDS	CDS	PLS
u	$5.460\,5\times 10^{-3}$	$1.579\,8\times 10^{-2}$	$5.308\,0\times 10^{-3}$	$2.703\,7\times 10^{-2}$
v	$3.891\,2\times 10^{-3}$	$2.871\,7\times 10^{-2}$	$3.943\,7\times 10^{-3}$	$4.660\,5\times 10^{-2}$
w	$1.603\,0\times 10^{-3}$	$5.535\,2\times 10^{-3}$	$1.537\,8\times 10^{-3}$	$4.911\,2\times 10^{-2}$

5 CONCLUSIONS

A perturbation finite volume (PFV) method is developed in the paper. Similar to the standard three-point schemes, for the two-dimensional case, the PFV scheme only uses four nodes for the triangle grids and five nodes for the Cartesian grids. The PFV scheme is an upwind and mixed scheme using any higher-order interpolation and second-order integration approximations. The PFV schemes are applied on a number of 1-D linear and nonlinear problems, 2-D and 3-D flow model equations. In general, with the order of interpolation greater than six, the PFV scheme produces an absolute maximum errors nearly unchanged for all problems. Comparing with the standard three-point schemes, a higher accuracy is obtained. The formulation of the PFV scheme is consistent with that of the first-order upwind scheme (UDS), however, its numerical diffusion is much smaller than the first-order upwind scheme. The calculations also show that the numerical accuracies of the PFV schemes are also much higher than those of the second-order central scheme (CDS), QUICK and power-law schemes. Thus the new three-point PFV scheme has higher accuracy and is especially suitable for the calculation on the unstructured grid for an arbitrary geometrical shape. Our next objective is to use the developed PFV scheme for the incompressible and compressible engineering flow problems.

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APPENDIX A

In this appendix, we prove that the present scheme (11) can be transformed into an exact numerical solution scheme of the following convective-diffusion differential equation

$$\rho(\mathbf{u} \cdot \nabla)\varphi = \Gamma \nabla^2 \varphi + q \quad (\text{A1})$$

The differential equation (A1) is the counterpart of the integral Eq.(1). Letting $N \rightarrow \infty$ in the formulae (11.1) and (11.2), the difference scheme (11) can be transformed into

$$\begin{aligned} & \frac{\Gamma \Delta y}{\Delta x \bar{G}_{px}} \{ [(1 + \text{sign } R_{\Delta x}) + (1 - \text{sign } R_{\Delta x})e^{-R_{\Delta x}}] \varphi_{i+1,j} - \\ & 2(1 + \cosh R_{\Delta x} + \text{sign } R_{\Delta x} \cdot \sinh R_{\Delta x}) \varphi_{i,j} + \\ & [(1 - \text{sign } R_{\Delta x}) + (1 + \text{sign } R_{\Delta x})e^{R_{\Delta x}}] \varphi_{i-1,j} \} + \\ & \frac{\Gamma \Delta x}{\Delta y \bar{G}_{py}} \{ [(1 + \text{sign } R_{\Delta y}) + (1 - \text{sign } R_{\Delta y})e^{-R_{\Delta y}}] \varphi_{i,j+1} - \\ & 2(1 + \cosh R_{\Delta y} + \text{sign } R_{\Delta y} \cdot \sinh R_{\Delta y}) \varphi_{i,j} + \\ & [(1 - \text{sign } R_{\Delta y}) + (1 + \text{sign } R_{\Delta y})e^{R_{\Delta y}}] \varphi_{i,j-1} \} + q_{i,j} = 0 \end{aligned} \quad (\text{A2})$$

where $\bar{G}_{px} = \sinh R_{\Delta x} + (\cosh R_{\Delta x} - 1)\text{sign } R_{\Delta x}$

$$\bar{G}_{py} = \sinh R_{\Delta y} + (\cosh R_{\Delta y} - 1)\text{sign } R_{\Delta y}$$

Now we prove that the basic and special solutions of the difference equation (A2) are the same as those of the linear differential equation (A1). The basic and special solutions of the differential equation (A2) are, respectively,

$$C, \left[\frac{1 - \text{sign } R_{\Delta x} + (1 + \text{sign } R_{\Delta x})e^{R_{\Delta x}}}{1 + \text{sign } R_{\Delta x} + (1 - \text{sign } R_{\Delta x})e^{-R_{\Delta x}}} \right]^i, \left[\frac{1 - \text{sign } R_{\Delta y} + (1 + \text{sign } R_{\Delta y})e^{R_{\Delta y}}}{1 + \text{sign } R_{\Delta y} + (1 - \text{sign } R_{\Delta y})e^{-R_{\Delta y}}} \right]^j \quad (\text{A3})$$

$$q_{i,j} \left(\frac{x_i}{2\rho u} + \frac{y_j}{2\rho v} \right) \quad (\text{A4})$$

where C is a constant. The constant C and the solution (A4) are obviously the basic and special solutions of the linear differential equation (A1). Now we consider the second basic solution of (A3). Fixing $x_i = i\Delta x$, $y_j = j\Delta y$, letting $\Delta x \rightarrow 0$ and $\Delta y \rightarrow 0$, the limit of the second basic solution is

$$\begin{aligned}
& \lim_{\substack{\Delta x \rightarrow 0 \\ \Delta y \rightarrow 0}} \ln \left\{ \left[\frac{1 - \operatorname{sign} R_{\Delta x} + (1 + \operatorname{sign} R_{\Delta x})e^{R_{\Delta x}}}{1 + \operatorname{sign} R_{\Delta x} + (1 - \operatorname{sign} R_{\Delta x})e^{-R_{\Delta x}}} \right]^{x_i/\Delta x} \left[\frac{1 - \operatorname{sign} R_{\Delta y} + (1 + \operatorname{sign} R_{\Delta y})e^{R_{\Delta y}}}{1 + \operatorname{sign} R_{\Delta y} + (1 - \operatorname{sign} R_{\Delta y})e^{-R_{\Delta y}}} \right]^{y_j/\Delta y} \right\} \\
& = \frac{\rho u}{F} x_i + \frac{\rho v}{F} y_j
\end{aligned} \tag{A5}$$

This is a basic solution of the linear differential equation (A1). Therefore, the scheme (A2) is an exact numerical solution scheme for the linear connective-diffusion

equation (A1). The above conclusion for the 2-D exact numerical solution scheme also holds for both the 1-D and 3-D problems.