Lateral Buckling of Interconnects in a Noncoplanar Mesh Design for Stretchable Electronics

Analytical models have been established to study the lateral buckling of interconnects under shear in a noncoplanar mesh design for stretchable electronics. Analytical expressions are obtained for the critical load and buckling shape at the onset of buckling by solving the equilibrium equations. The postbuckling behavior is studied by energy minimization of the potential energy, including up to fourth power of the displacement. A simple expression of the amplitude characterizing the deformation after buckling is obtained. These results agree well with the finite element simulations without any parameter fitting. The models in this paper may provide a route to study complex buckling modes of interconnects, such as diagonal compression/stretching involving both compression and shear. [DOI: 10.1115/1.4023036]

Keywords: lateral buckling, finite element analysis, energy method

1 Introduction

Stretchable electronics is an emerging technology that enables an electronic system with large stretchability. It has various applications, including flexible displays [1], electronic eye cameras [2–4], biointegrated devices [5–7], electronic sensors for robotics [8,9], and structural health-monitoring devices [10]. Organic semiconductor materials can sustain relatively large deformation and have been used to develop stretchable electronics, but their electrical performance is relatively poor comparing to conventional inorganic semiconductor materials. Compatibility with well-developed, high-performance inorganic semiconductor materials represents a key advantage in this area. The challenge is to make these inorganic semiconductor materials stretchable without sacrificing the electrical performance, since they are brittle and cannot sustain large deformation. One successful approach is to use noncoplanar mesh design [11] on a compliant substrate, which is based on the interconnect-island concept.

Figure 1(a) schematically shows the noncoplanar mesh design, where the islands are chemically bonded to the compliant substrate while the interconnects are loosely bonded. The interconnects between islands can buckle to accommodate the external strains. When the compression (or the release of the prestretch in the substrate) is applied along the interconnect direction, the interconnects undergo Euler buckling, which has been studied thoroughly [12]. When shear (Fig. 1(b)) or diagonal stretch (45 deg from the interconnects) is applied, the interconnects may undergo lateral buckling. Different from Euler buckling, where the displacement takes the simple sinusoidal form, lateral buckling involves large torsion and out-of-plane bending with a very complex form of displacements. The solution of lateral buckling is important to guide the design of stretchable electronics. Most of existing results on lateral buckling [13,14], however, are based on numerical methods, and closed-form solutions only exist for initial buckling, but not for postbuckling. Recently, Su et al. [15] established a systematic method based on the nonlinear equilibrium equations for postbuckling of beams that may involve rather complex buckling modes, such as later buckling. The perturbation method is used to obtain the amplitude of buckled interconnects. The objective of this paper is to derive analytical solutions for...
lateral buckling (initial buckling and postbuckling) of interconnects under shear. The initial buckling of interconnects is investigated through the equilibrium equations to find the critical buckling load and buckling shape. The postbuckling behavior is studied by energy minimization of the potential energy, including up to fourth power of the displacement. The paper is outlined as follows: The initial buckling analysis is described in Sec. 2, while the postbuckling analysis is presented in Sec. 3. Section 4 describes the finite element simulations, and the results are discussed in Sec. 5.

2 Initial Buckling Analysis

2.1 Equations for Initial Buckling. The interconnect can be modeled as an L-long beam with a thin rectangular cross-section $a \times b$, where $b \gg a$. The left end is fixed, and the right end is subjected to a shear or vertical displacement $v_0$ with no rotations (Fig. 2) (due to the rigid island). To investigate the initial lateral buckling of the beam, we start from the equilibrium equations [16]. Two sets of coordinate axes $(x, y, z)$ and $(\xi, \eta, \zeta)$ are defined as shown in Fig. 2. The first set of coordinate axes $(x, y, z)$ is fixed in space with the axis $z$ coinciding with the centroidal axis of the undeformed beam and the $x$ and $y$ coinciding with the principal axes of the cross-section. The second is a set of moving axes $(\xi, \eta, \zeta)$ fixed to a point on the centroidal axis of the beam and moves with it. The axes $\xi$ and $\eta$ are the principal axes of the cross-section on the deformed beam, and $\zeta$ coincides with the tangent to the deformed centroidal axis of the beam. The deformation of the beam is defined by the components $u$, $v$, and $w$ of the displacement of the center in the $x$, $y$, and $z$ directions and by the rotation $\phi$ of the cross-section (Fig. 2). Components $u$ and $v$ are taken positive in the positive direction of the corresponding axes, and the angle $\phi$ is taken positive about the $z$ axis according to the right-hand rule.

The bending moments can be easily obtained as

$$
\begin{align*}
M_x &= p_z \\
M_y &= m_y + p_z \left( \frac{L}{2} - z \right) \\
M_z &= t - p_y u + p_x v
\end{align*}
$$

(1)

where $p_z$ denotes the load in the $y$ direction to cause $v_0$ on the right end, $p_x$ the load in the $x$ direction on the right end, $m_y$ the bending moment in the $y$ direction on the right end, and $t$ the torque in the $z$ direction on the right end. The following orders of forces and displacement hold during buckling: the applied force and displacement (i.e., $p_x$, $m_y$, $t$, $u$, and $v$) result from lateral buckling are first order [15]. Projecting the above moments to the axes $(\xi, \eta, \zeta)$ and neglecting any terms higher than the second order yields

$$
\begin{align*}
M_\xi &= p_z \\
M_\eta &= -p_z \bar{\phi} + m_y + p_z \left( \frac{L}{2} - z \right) \\
M_\zeta &= p_y \frac{du}{dz} + p_x \frac{dv}{dz} \left( \frac{L}{2} - z \right) + t - p_y u + p_x v
\end{align*}
$$

(2)

The deflected beam’s constitutive equations are

$$
\begin{align*}
\frac{d^2 v}{dz^2} &= -M_\xi \\
\frac{d^2 u}{dz^2} &= M_\eta \\
C \frac{d\phi}{dz} &= M_\zeta
\end{align*}
$$

(3)

where $C = Ea/b/[6(1 + \nu)]$ is the torsional rigidity, $E$ is the Young’s modulus, and $\nu$ is the Poisson’s ratio, and $I_\xi = ab^3/12$ and $I_\eta = a^3 b/12$ are the moment of inertia about axes $\xi$ and $\eta$, respectively. The boundary conditions are

$$
\begin{align*}
\begin{cases}
&u(-L/2) = u'(L/2) = u'(L/2) = 0 \\
v(-L/2) = v'(L/2) = v'(L/2) = 0 \\
&\phi(-L/2) = \phi(L/2) = 0
\end{cases}
\end{align*}
$$

(4)

By introducing a nondimensional coordinate $s = 2z/L$, where $-1 \leq s \leq 1$, Eqs. (3) and (4) become

$$
\begin{align*}
\frac{d^2 v}{ds^2} &= -\left( \frac{L}{2} \right)^3 p_z s \\
\frac{d^2 u}{ds^2} &= \left( \frac{L}{2} \right)^3 \left[ -p_z \bar{\phi} + \frac{2}{L} m_y + p_z (1 - s) \right] \\
C \frac{d\phi}{ds} &= \frac{L}{2} \left[ p_z \frac{du}{ds} + \frac{dv}{ds} \right] (1 - s) + t - p_y u + p_x v
\end{align*}
$$

(5)

and

$$
\begin{align*}
\begin{cases}
&u(-1) = u'(1) = u'(1) = 0 \\
v(-1) = v'(1) = v'(1) = 0 \\
&\phi(-1) = \phi(1) = 0
\end{cases}
\end{align*}
$$

(6)

After eliminating $u$ from the second and third equations in Eq. (5) and applying $I_\zeta \gg I_\eta$, we obtain

$$
\frac{d^2 \phi}{ds^2} + \frac{p_z^2}{CET_\eta} \left( \frac{L}{2} \right)^4 s^2 \phi = \left( \frac{L}{2} \right)^4 \frac{p_y}{CET_\eta} \left[ \frac{2}{L} m_y + p_z (1 - s) \right]
$$

(7)

The solution $\phi$ of Eq. (7) is either symmetric or antisymmetric.

2.2 Symmetric Buckling Mode. The symmetric buckling mode corresponds to an even function for $\phi$ and an odd function for $u$. Integration of the second part of Eq. (5) from $-1$ to $1$ gives $2m_y/L + p_z = 0$. Then, Eq. (7) becomes

$$
\frac{d^2 \phi}{ds^2} + \frac{p_z^2}{CET_\eta} \left( \frac{L}{2} \right)^4 s^2 \phi = 0
$$

(8)

The solution $\phi$ of Eq. (8) is either symmetric or antisymmetric.
\[ \frac{d^2 \phi}{ds^2} + \frac{p_s^2}{CEI_y} \left( \frac{L}{2} \right)^4 s^2 \phi = -\left( \frac{L}{2} \right)^4 p_s \beta_s \left( \frac{BL^2}{8} \right)^2 \]  

(8)

The above equation can be solved as

\[ \phi(s) = \sqrt{\frac{2}{s}} \left[ AJ_{-1/4} \left( \frac{BL^2}{8} \right)^2 \right] + BJ_{1/4} \left( \frac{BL^2}{8} \right)^2] - \frac{p_s}{\beta_s} p_y \]

(9)

where \( A \) and \( B \) are constants to be determined by boundary conditions, \( J \) is the Bessel function of the first kind, and \( \beta = \sqrt{\frac{2p_y^2}{EI_yC}} \). After applying the boundary conditions \( \phi'(0) = 0 \) and \( \phi(1) = 0 \), \( \phi(s) \) in Eq. (9) becomes

\[ \phi(s) = A \left[ \sqrt{\frac{2}{s}} J_{-1/4} \left( \frac{BL^2}{8} \right)^2 \right] - J_{1/4} \left( \frac{BL^2}{8} \right)^2] \]

(10)

The displacement \( u(s) \) can then be obtained from the second part of Eq. (5) as

\[ u(s) = -\frac{Ap_s}{EI_y} \left( \frac{L}{2} \right)^2 \int_0^1 \left[ \int_{-1}^1 \sqrt{\frac{2}{s}} J_{-1/4} \left( \frac{BL^2}{8} \right)^2 \right] ds \]

(11)

The boundary \( u(1) = 0 \) yields

\[ J_{3/4} \left( \frac{BL^2}{8} \right)^2 = 0 \]

(12)

which can be solved for \( L^2 \beta_s / 8 \). The critical load for lateral buckling is then given by

\[ p_s^{cr} = \sqrt{EI_yC} \beta_s = \frac{Ep_s^2}{6} \frac{1}{\sqrt{2(1+\nu)}} \beta_s \]

(13)

The displacement \( v(s) \) can be obtained from the first part of Eq. (5) and second part of Eq. (6) as

\[ v(s) = -\frac{p_s L^3}{48EI_y} \left( s^3 - 3s - 2 \right) \]

(14)

The corresponding critical displacement \( v_0^{cr} \) is then given by

\[ v_0^{cr} = \frac{L^2 p_s^{cr}}{12EI_y} = \frac{L^2 \beta_s^{cr}}{6b^2} \sqrt{\frac{1}{2(1+\nu)}} \]

(15)

By introducing a nondimensional parameter, \( \alpha = \beta_s L^2 \), the buckling shape is characterized by \( \phi(s) \) in Eq. (10) and \( u(s) \) in Eq. (11), which become

\[ \phi(s) = A\phi(s), \quad \text{and} \quad u(s) = -\frac{L}{2} \sqrt{\frac{1}{2(1+\nu)}} \bar{u}(s) \]

(16)

where \( \phi(s) \) and \( \bar{u}(s) \) are independent of the materials and geometries, and they only depend on the nondimensional coordinate \( s \) as

\[ \phi(s) = \sqrt{\frac{2}{s}} J_{-1/4} \left( \frac{BL^2}{8} \right)^2 \] \( \quad \text{and} \quad u(s) = \alpha \left[ \int_{-1}^1 \sqrt{\frac{2}{s}} J_{-1/4} \left( \frac{BL^2}{8} \right)^2 \right] ds \]

(17)

We refer \( A \) as the amplitude in this paper, which can be obtained by postbuckling analysis in Sec. 3.

### 2.3 Antisymmetric Buckling Mode

The antisymmetric buckling mode corresponds to an odd function for \( \phi \) and an even function for \( u \). From Eq. (7), \( p_s = 0 \). Then, Eq. (7) becomes

\[ \frac{d^2 \phi}{ds^2} + \frac{p_s^2}{CEI_y} \left( \frac{L}{2} \right)^4 s^2 \phi = \frac{L^3}{2} \beta_s \phi \]

(18)

The solution of Eq. (19) is obtained as

\[ \phi(s) = \sqrt{\frac{2}{s}} \left[ AJ_{1/4} \left( \frac{BL^2}{8} \right)^2 \right] + BJ_{-1/4} \left( \frac{BL^2}{8} \right)^2] + \frac{L}{2} \left[ \int_{-1}^1 s^3 \phi \left( \frac{BL^2}{8} \right)^2 \right] ds \]

(19)

where \( \phi_p(x) \) is

\[ \phi_p(x) = -\frac{1}{48s^2} \left[ 2s J_{1/4}(x)J_{-1/4}(x) \right] + \frac{1}{2} \left[ J_{1/4}(x)J_{-1/4}(x) \right] \left\{ \left[ J_{1/4}(x)J_{-1/4}(x) \right] \right\} \]

(20)

From \( \phi(1) = 0 \), we have

\[ J_{1/4} \left( \frac{BL^2}{8} \right)^2 + \frac{4}{1} \left( \frac{BL^2}{8} \right)^2 \int_{-1}^1 s^3 \phi \left( \frac{BL^2}{8} \right)^2 ds \phi_p \left( \frac{BL^2}{8} \right)^2 = 0 \]

(21)

which can be solved for \( L^2 \beta_s / 8 \). The critical load in Eq. (13) and critical displacement in Eq. (15) for symmetric mode still hold for antisymmetric mode. The buckling shape can also be given by Eq. (16), except \( \phi(s) \) and \( \bar{u}(s) \) are different and given by
\[ \hat{\phi}(s) = \sqrt{3} J_{1/4} \left( \frac{2}{5} z^2 \right) + \frac{4}{1 - 4} \left( \frac{2}{5} \right) \int_{z=0}^{s} J_{1/4} \left( \frac{2}{5} \xi^2 \right) \right] ds^2 \phi_p \left( \frac{2}{5} s^2 \right) \]

(23)

\[ \hat{u}(s) = \pi \int_{l=-\infty}^{l=\infty} \left[ \xi \sqrt{3} J_{1/4} \left( \frac{2}{5} s^2 \right) \right] - \left[ 1 - \frac{z^2}{16} \xi^4 \phi_p \left( \frac{2}{5} s^2 \right) \right] \int_{l=0}^{l=\infty} \gamma \sqrt{3} J_{1/4} \left( \frac{2}{5} \gamma^2 \right) \right] \right] ds \]

(24)

3 Postbuckling Analysis

The postbuckling behavior is studied by energy minimization of the potential energy of the beam. The displacements due to buckling at neutral axis are \( u(z) \), \( v(z) \), and \( w(z) \). Then, the general point at a section has the following displacements by ignoring the warping as \[ \begin{align*}
\{ u_1(x, y, z) &= u(z) - \phi(y) \\
v_1(x, y, z) &= v(z) + \phi(x) \\
w_1(x, y, z) &= w(z) - \phi(x) - \phi(y) 
\end{align*} \]

(25)

The longitudinal and shear strains due to buckling can be calculated by

\[ \left\{ \begin{align*}
\varepsilon_z &= \frac{\partial u_1}{\partial z} + \frac{1}{2} \left( \frac{\partial u_1}{\partial x} \right)^2 + \frac{1}{2} \left( \frac{\partial u_1}{\partial y} \right)^2 + \frac{1}{2} \left( \frac{\partial u_1}{\partial z} \right)^2 \\
\varepsilon_y &= \frac{1}{2} \left( \frac{\partial u_1}{\partial z} + \frac{\partial u_1}{\partial y} \right)
\end{align*} \]

(26)

The strain energy of the beam can be obtained by \[ \begin{align*}
U_{\text{strain}} &= \frac{1}{2} \varepsilon_0 \varepsilon_z + \frac{1}{2} E \varepsilon_z \varepsilon_z \right] ds + \frac{1}{2} C \int_{l=0}^{l=\infty} \left( \frac{d \phi}{dz} \right)^2 dz
\end{align*} \]

(27)

where \( \sigma_0 = p_{yz} / I_z \) is the initial normal stress in the cross-section due to bending, \( \sigma_y = p_y / (ab) \) is the initial shear stress, \( p_y = \sqrt{E} C \beta_{yz} \), \( C \) is the torsional rigidity, and \( V \) is the volume of the beam. The displacement \( v(z) \) is negligible due to buckling. The force equilibrium in \( z \) direction \( \int_{l=0}^{l=\infty} E \varepsilon_z a ds = 0 \) gives \( w' = -\frac{1}{2} \beta_y^2 - \frac{1}{2} \phi^2 (a^2 + b^2) \). Substituting Eq. (16) to the energy expression in Eq. (27) and ignoring the terms higher than \( A^2 \), the energy then becomes

\[ \begin{align*}
U_{\text{strain}} &= A^2 \frac{E a^2 b}{96(1 + \nu) L} \left[ - \frac{24 a b^2 v_0}{a^2 L} \sqrt{\frac{1 + \nu}{2}} (C_1 + C_3) + 16 C_6 \\
&+ \frac{\alpha^2 C_1}{a} \right] + A^2 \frac{E a b^2}{8 L^2} \left[ \frac{7}{45} C_2 + \frac{\alpha^2 L^2}{12(1 + \nu) b^2} C_4 \right] \right]
\end{align*} \]

(28)

4 Finite Element Element

The finite element element method is also used to study the lateral buckling of the beam in order to validate the analytical solutions. The element B33 in ABAQUS is used to discretize the beam with left end fixed and right end subjected to a vertical displacement. We first determine the eigenvalues and eigenmodes from the buckling analysis. The eigenmode is then used as initial small geometrical imperfection to trigger the buckling in the postbuckling analysis. It should be noted that the imperfections are always small enough to ensure that the results are independent of the imperfections.

5 Results and Discussion

We take a beam with \( L = 20 \), \( a = 1 \), \( b = 0.1 \), and \( \nu = 0.3 \) as an example to show our results. Equations (12) and (15) give the critical displacement for the symmetrical buckling modes. The first two roots of Eq. (22) are obtained as \( L^2 (\beta_{yz})^2 / 8 = 3.491 \) and \( L^2 (\beta_{yz}) / 8 = 6.653 \), which yield the critical displacements \( \nu_{cr} = 0.5773 \) and \( 1.1002 \) for the first and second symmetrical buckling modes, respectively. Similarly, Eqs. (15) and (22) give the critical displacement for the antisymmetric buckling modes. The first two roots of Eq. (22) are \( L^2 (\beta_{yz}) / 8 = 4.6146 \) and \( L^2 (\beta_{yz}) / 8 = 6.0476 \), which yield the critical displacements \( \nu_{cr} = 0.7632 \) and \( 1.0001 \). According to the magnitude of critical displacement, mode 1 for lateral buckling is symmetric, mode 2 is antisymmetric, mode 3 is antisymmetric, and mode 4 is symmetric. The buckling shapes can then be obtained by Eq. (17) for symmetric mode and Eqs. (23) and (24) for antisymmetric mode. Table 1 shows the critical displacements for these four modes of lateral buckling, which are compared to those from eigenvalue buckling analysis in the finite element software ABAQUS [19]. The analytical predictions agree well with numerical simulations with an error less than 3%. The buckling shapes of rotation of the cross-section and displacement of center in \( x \) direction, \( \phi(s) \) and \( \hat{u}(s) \), are shown in Figs. 3 and 4, respectively. The results from finite element simulations are also shown for comparison and validation. The analytical predictions have good agreements with the finite element simulations (dot).

<table>
<thead>
<tr>
<th>Mode 1</th>
<th>Mode 2</th>
<th>Mode 3</th>
<th>Mode 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \nu_{cr}^{\text{analytical}} )</td>
<td>0.5773</td>
<td>0.7632</td>
<td>1.0001</td>
</tr>
<tr>
<td>( \nu_{cr}^{\text{FEM}} )</td>
<td>0.5613</td>
<td>0.7420</td>
<td>0.9732</td>
</tr>
</tbody>
</table>
Let us consider the postbuckling for mode 1 with \( L^2 \beta / 8 = 3.491 \). \( \alpha = \beta \alpha L^2 \) is then obtained as 27.9280. The constants \( C_1 - C_6 \) can be calculated from Eq. (29) as given by 
\[
C_1 = 6.323e^{-2}, \quad C_2 = 2.186e^{-1}, \quad C_3 = 8.997e^{-2}, \quad C_4 = 1.013e^{-2}, \\
C_5 = 2.682e^{-2}, \quad \text{and} \quad C_6 = 4.390. 
\]
The amplitude \( A \) in Eq. (30) then becomes
\[
A = 1.3336 \frac{1}{\sqrt{L}} \left( \frac{3.4001 b^2}{L^2} + \frac{0.6584}{1 + \nu} \right)^{1/2} \sqrt{1 + \nu} \cdot \sqrt{1 - 0.32905 \frac{a^2 L}{b^2 \sqrt{1 + \nu}}} 
\]
For \( b \ll L \), if \( b \ll L \), \( A \) can be simplified as
\[
A = 1.6435 \sqrt{1 + \nu} \cdot \sqrt{1 - 0.32905 \frac{1}{L} \frac{a^2}{b^2}} 
\]
Figure 5 shows the amplitude \( A \) versus the applied displacement \( v_0 \) for the buckling mode 1 of a beam with \( L = 20, \ a = 1, \ b = 0.1, \) and \( \nu = 0.3 \). The amplitude \( A \) remains zero when \( v_0 \) is smaller than a critical value, \( v_{0, \text{cr}}^2 = 3.2905a^2L/(\sqrt{1 + \nu}) = 0.5773 \), and buckling does not occur. This critical value is consistent with that (Eq. (15)) from initial buckling analysis. Once \( v_0 \) exceeds the critical value, lateral buckling occurs, and the amplitude increases as \( v_0 \) increases. To validate the analytical solution, a postbuckling analysis is performed in ABAQUS. The buckling modes are first obtained through the eigenvalue buckling analysis and then used as initial small geometrical imperfections to trigger the buckling of the interconnect. It should be noted that the imperfection should be small enough to ensure that the solution is accurate. As shown in Fig. 5, the finite element results agree well with analytical solutions.

6 Concluding Remarks

We have derived analytical solutions for lateral buckling of interconnects under shear in a noncoplanar mesh design for stretchable electronics. The critical buckling load and mode are obtained analytically by solving the equilibrium equations, and they agree well with finite element simulations. The postbuckling behavior is studied by energy minimization of the potential energy, including up to fourth power of the displacement. A simple expression of the amplitude to characterize the magnitude of deformation is obtained, and it agrees well with the finite element simulations without any parameter fitting. The models in this paper may provide a route to study complex buckling modes of interconnects, such as diagonal compression/stretching involving both compression and shear.

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