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An analytical mechanics model for the island-bridge structure of stretchable electronics

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In stretchable electronics, the island-bridge structure on a soft substrate plays an important role in achieving large stretchability. A key issue in developing such a system is to prevent the island-bridge structure from breaking during use because it is composed of brittle semiconductor materials (*e.g.*, silicon) which withstand very small strains (∼1%). In the following, an analytical mechanics model of the island-bridge structure is established and the accurate solution is obtained. A validated scaling law is found to reveal the dependence of the normalized maximum strain in the island on the prestrain of the substrate, which controls the mechanical failure of the island-bridge structure and provides a theoretical basis for fracture-safe design of stretchable electronics.

1 Introduction

Stretchable electronics[1–5] serves as a promising technology with applications to many emerging devices such as flexible displays[6] and solar cells,[7] conformable sensors,[8] stretchable LEDs,[9] implantable transient electronics[10,11] and electronic eye cameras.[12,13] Rapid development of the field of stretchable electronics has attracted much interest in modeling, design and fabrication of such structures and devices.[14–20] Until now, stretchability of the electronics has been achieved at different levels by several approaches. The coplanar stretchable interconnects between rigid devices over the compliant polymer substrate were developed to make the elastic circuit, with both the islands and interconnects bonded to the substrate.[21] The circuit designed by this approach remains functional when stretched and relaxed by around 10% strain. Another impressive approach is the wavy configuration of the circuit itself.[22] Some degree of stretchability (*e.g.*, 10%) could be reached in this way, yet the mechanical safety and electronic performance of the devices under large strains are still not guaranteed. A novel island-bridge structure[23] was designed to achieve large stretchability for stretchable electronics by a noncoplanar mesh configuration[12,15,19] while maintaining their high electronic performance. The fabrication process of such a structure involves the transfer printing[18,24] of "island-like" semiconductor devices (*e.g.*, silicon), connected with each other by planar "bridge-like" metal interconnects, onto a biaxially prestrained elastometric substrate (*e.g.*, PDMS), and the release of prestrain which causes the bridges to lift vertically off the substrate and forms the arc-shaped structures. Fig. 1 schematically shows this process by a finite element model. A scanning electron micrograph (SEM) of the stretchable silicon island-bridge structure on a PDMS substrate is shown in Fig. 2.[12]

It has been shown in the literature that a bridge can be modeled as a beam with both ends clamped and the model was solved *via* energy minimization by assuming a sinusoidal buckle profile for the bridge deformation.[12,15,19] However, this solution is reliable within the range of relatively small deflections of the bridge but is not accurate enough for large deflections which cannot be approximated well with the sinusoidal profile. For the deformation of an island, the finite element method (FEM) combined with the approximate dimensional analysis was used.[15] Since the island-bridge structure usually undergoes large deformation (∼100% applied strain), an accurate mechanics model to predict the strains in the system is critically needed, which will provide guidelines for the design of an island-bridge structure against mechanical failure. This paper aims at establishing an analytical model for the island-bridge structure by discarding the assumption of the sinusoidal buckle profile for the bridge deformation and the approximation of dimensional analysis for the island deformation.

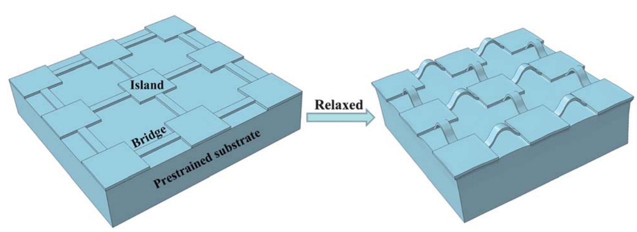

Fig. 1 Fabrication process of the island-bridge structure.

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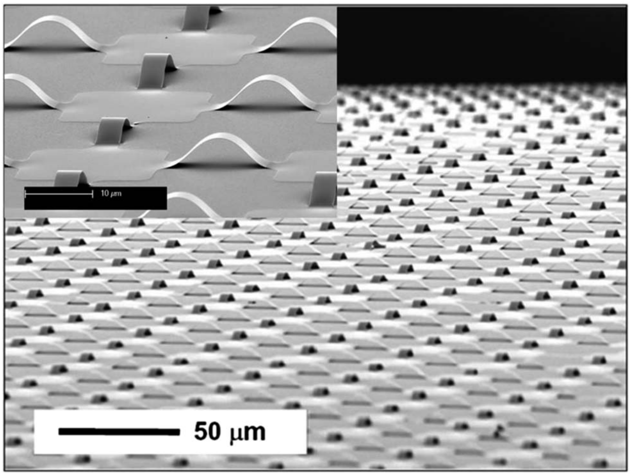

Fig. 2 Low and high magnification SEM images of the silicon island-bridge structure on a PDMS substrate. Reproduced with permission from ref. 12. ©2008, Nature Publishing Group.

A mechanics model of a buckled beam in large deflection, with the length change and rotation at the ends neglected,[12,15,19] is developed to better describe the deformation of the bridge. The island is modeled as a plate on an elastic substrate[25–27] with moments applied on its edges by the bridges. The model developed in this paper yields a scaling law to identify the non-dimensional combinations of material and geometry parameters that control the strain of the island. FEM is used to validate the scaling law, which is useful for the optimal design of the island-bridge structure and other film/substrate systems.

2 Mechanics model for the bridge structure

As shown in Fig. 3a, at a given prestrain ε_{pre} of the substrate, the bridge with length L_{bridge} buckles to accommodate the release of prestrain, which yields the distance $L_{\text{bridge}}/(1+\varepsilon_{\text{pre}})$ between the two ends A and E. In view of the symmetry, only part of the bridge with length $L_{\text{bridge}}/4$ is analyzed, as shown in Fig. 3b. This is indeed a problem of large deflection of a buckled bar (the elastica).[28]

The governing equation of the bridge is[28]

$$E_{\text{bridge}} I_{\text{bridge}} \frac{\mathrm{d}\theta}{\mathrm{d}s} = P(w_B - w) \tag{1}$$

where $E_{\text{bridge}} I_{\text{bridge}}$ is the flexural rigidity of the bridge, E_{bridge} is the Young's modulus, I_{bridge} is the moment of inertia, θ is the angle which the tangent at a point of the curved bridge AB makes with the x axis, s is the distance along the axis of the curved bridge from A, P is the force applied at B by the adjacent part of AB, w is the deflection in the z direction and w_B is the deflection at B.

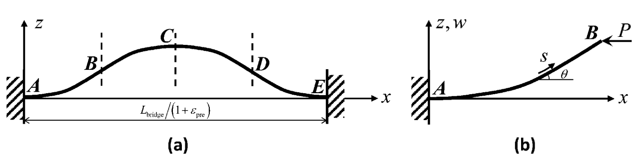

Fig. 3 Schematic illustration of the model for the buckled bridge by release of prestrain of the substrate.

Noting that $\mathrm{d}w/\mathrm{d}s = \sin\theta$, differentiating eqn (1) with respect to s gives

$$E_{\text{bridge}} I_{\text{bridge}} \frac{\mathrm{d}^2\theta}{\mathrm{d}s^2} = -P\sin\theta \tag{2}$$

Using the boundary conditions at the right end,

$$\theta = \alpha, \quad \frac{\mathrm{d}\theta}{\mathrm{d}s} = 0 \text{ at } B \tag{3}$$

the integral of eqn (2) yields

$$\frac{1}{2}\left(\frac{\mathrm{d}\theta}{\mathrm{d}s}\right)^2 = \lambda^2(\cos\theta - \cos\alpha) \tag{4}$$

where $\lambda^2 = P/(E_{\text{bridge}} I_{\text{bridge}})$.

For the present model, solving for $\mathrm{d}s$ from eqn (4) gives

$$\mathrm{d}s = \frac{\mathrm{d}\theta}{\lambda\sqrt{2(\cos\theta - \cos\alpha)}} \tag{5}$$

Introducing $\beta = \sin(\alpha/2)$, the total length of the bridge is obtained by eqn (5) as

$$L_{\text{bridge}} = 4\int_0^{L_{\text{bridge}}/4} \mathrm{d}s = \frac{4\tilde{K}(\beta)}{\lambda} \tag{6}$$

where $\tilde{K}(\beta)$ is the complete elliptic integral of the first kind. From eqn (6), $\lambda = 4\tilde{K}(\beta)/L_{\text{bridge}}$.

The coordinates of B at the curved bridge are

$$\begin{aligned} \tilde{x}^B_{\text{bridge}} &= \int_0^{L_{\text{bridge}}/4} \cos\theta \mathrm{d}s = \frac{L_{\text{bridge}}}{4}\left[\frac{2\tilde{E}(\beta)}{\tilde{K}(\beta)} - 1\right] \\ \tilde{z}^B_{\text{bridge}} &= \int_0^{L_{\text{bridge}}/4} \sin\theta \mathrm{d}s = \frac{\beta L_{\text{bridge}}}{2\tilde{K}(\beta)} \end{aligned} \tag{7}$$

where $\tilde{E}(\beta)$ is the complete elliptic integral of the second kind.

The maximum deflection of the bridge is

$$w^{\max}_{\text{bridge}} = 2\tilde{z}^B_{\text{bridge}} = \frac{\beta L_{\text{bridge}}}{\tilde{K}(\beta)} \tag{8}$$

The tensile prestrain of the substrate is

$$\varepsilon_{\text{pre}} = \frac{L_{\text{bridge}}\left(1 - \dfrac{4\tilde{x}^B_{\text{bridge}}}{L_{\text{bridge}}}\right)}{L_{\text{bridge}}\dfrac{4\tilde{x}^B_{\text{bridge}}}{L_{\text{bridge}}}} = \frac{1}{2\dfrac{\tilde{E}(\beta)}{\tilde{K}(\beta)} - 1} - 1 \tag{9}$$

The end moment of the bridge is

$$M_0 = P\tilde{z}^B_{\text{bridge}} = 8\beta\tilde{K}(\beta)\frac{E_{\text{bridge}} I_{\text{bridge}}}{L_{\text{bridge}}} \tag{10}$$

thus the maximum strain in the bridge is

$$\varepsilon^{\max}_{\text{bridge}} = \frac{t_{\text{bridge}}}{2}\frac{M_0}{E_{\text{bridge}} I_{\text{bridge}}} = 4\beta\tilde{K}(\beta)\frac{t_{\text{bridge}}}{L_{\text{bridge}}} \tag{11}$$

where t_{bridge} is the bridge thickness.

Eqn (8)–(11) are the parametric equations with respect to β for $w^{\max}_{\text{bridge}}$, ε_{pre}, M_0 and $\varepsilon^{\max}_{\text{bridge}}$. At a given prestrain ε_{pre}, β is

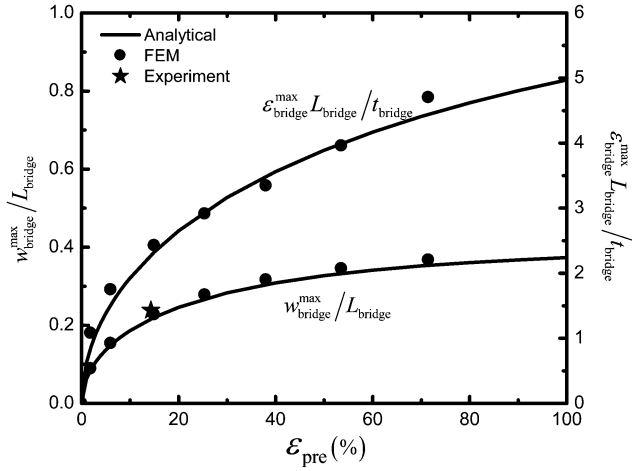

Fig. 4 Normalized maximum deflection and maximum strain of the bridge *versus* the prestrain of the substrate.

obtained from eqn (9), substituting which into eqn (8), (10) and (11) gives w^{max}_{bridge}, M_0 and $\varepsilon^{max}_{bridge}$. M_0 is applied to the island as the external force.

Eqn (8), (10) and (11) suggest that the maximum deflection w^{max}_{bridge}, bending moment at the end M_0 and maximum strain of the bridge $\varepsilon^{max}_{bridge}$, normalized by L_{bridge}, $E_{bridge}I_{bridge}/L_{bridge}$ and t_{bridge}/L_{bridge}, respectively, depend only on the prestrain ε_{pre} of the substrate *via* eqn (9). The results are shown in Fig. 4 and agree well with those from FEM and the experimentally measured maximum deflection (4.76 μm for 14.3% prestrain).[12,15] In the analytical model and FEM simulation,[29] the material and geometry parameters are the same as those from the experiment[12,15] for 20 × 20 μm^2, 50 nm thick silicon islands and 20 × 4 μm^2, 50 nm thick silicon bridges on a 1 mm thick PDMS substrate, with the Young's moduli $E_{bridge} = E_{island} =$ 130 GPa and $E_{substrate} =$ 2 MPa, and Poisson's ratios $\nu_{bridge} = \nu_{island} = 0.27$ and $\nu_{substrate} = 0.48$.[12,15] In FEM, the islands and bridges are modeled by the shell element S4R since they are much thinner than the substrate which is characterized by the solid element C3D8R. The island and bridge layers are mounted on the prestretched substrate. The share-node technique is used to bond the islands and the substrate, and the bridge–substrate interaction is hard contact. After the prestrain in the substrate is released, the bridge layer will buckle, as shown in Fig. 1.

w^{max}_{bridge}, M_0 and $\varepsilon^{max}_{bridge}$ are proportional to L_{bridge}, $E_{bridge}I_{bridge}/L_{bridge}$ and t_{bridge}/L_{bridge}, respectively, which suggests that, while increasing the maximum deflection, a thin and long bridge helps to reduce the maximum strain and the moment intensity at the end thus increases the stretchability.

3 Mechanics model for the island structure

In a unit cell of the island-bridge system, the four buckled bridges impose bending moments and axial forces to the island at the four edges, where the strains due to the axial forces are negligible as compared to those due to the bending moments.[12,15,19] In the following, an analytical mechanics model is established which describes the island structure by a plate on a Winkler-type substrate equivalent to an elastic half-space[25–27] subjected to the bending moments of intensity M_0 over the bridge width b at the four edges of the island.

3.1 Governing equations for deformation of the island structure

The island has the edge length a and thickness t_{island}, and the origin of the coordinate system (x,y,z) is at the corner of the island midplane, with x and y along the island edges and z pointing into the substrate, as shown in Fig. 5a.

The equilibrium equations of the island are

$$\frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} - Q_x = 0, \quad \frac{\partial M_y}{\partial y} + \frac{\partial M_{xy}}{\partial x} - Q_y = 0,$$
$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} - k w_{island} = 0 \qquad (12)$$

and the internal forces are

$$M_x = -D\left(\frac{\partial^2 w_{island}}{\partial x^2} + \nu_{island}\frac{\partial^2 w_{island}}{\partial y^2}\right),$$
$$M_y = -D\left(\frac{\partial^2 w_{island}}{\partial y^2} + \nu_{island}\frac{\partial^2 w_{island}}{\partial x^2}\right),$$
$$M_{xy} = -D(1-\nu_{island})\frac{\partial^2 w_{island}}{\partial x \partial y}$$
$$Q_x = -D\frac{\partial(\nabla^2 w_{island})}{\partial x}, \quad Q_y = -D\frac{\partial(\nabla^2 w_{island})}{\partial y},$$
$$V_x = Q_x + \frac{\partial M_{xy}}{\partial y}, \qquad V_y = Q_y + \frac{\partial M_{xy}}{\partial x} \qquad (13a\text{–}g)$$

where k is the foundation modulus of the substrate, w_{island} is the deflection of the island in the z direction, $D = E_{island}t_{island}{}^3/[12(1 - \nu_{island}{}^2)]$ is the flexural rigidity, M_x and M_y are the bending moments, M_{xy} is the torsional moment, Q_x and Q_y are the shear forces, and V_x and V_y are the equivalent shear forces, respectively.

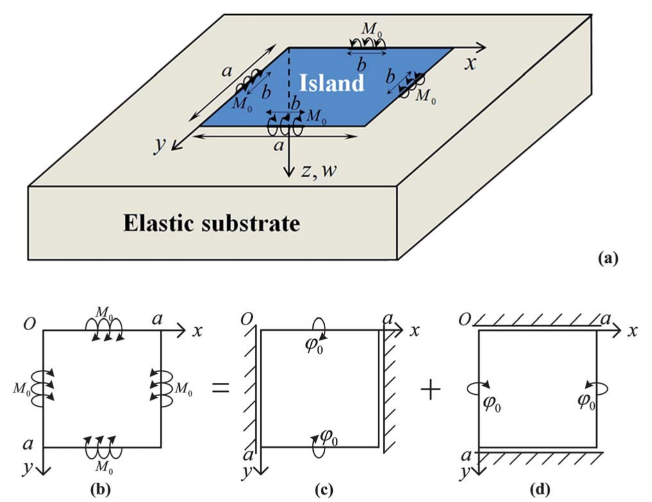

Fig. 5 Schematic illustration of the analytical model for the loaded island on a substrate.

Eqn (12) and (13a–g) can be transformed into the Hamiltonian canonical equations,[30] which are represented in the matrix form as

$$\frac{\partial \boldsymbol{Z}}{\partial y} = \boldsymbol{HZ} \tag{14}$$

where $\boldsymbol{H} = \begin{bmatrix} \boldsymbol{F} & \boldsymbol{G} \\ \boldsymbol{Q} & -\boldsymbol{F}^T \end{bmatrix}$,

$\boldsymbol{Q} = \begin{bmatrix} D({\nu_{\text{island}}}^2 - 1)\partial^4/\partial x^4 - k & 0 \\ 0 & 2D(1 - \nu_{\text{island}})\partial^2/\partial x^2 \end{bmatrix}$,

$\boldsymbol{F} = \begin{bmatrix} 0 & 1 \\ -\nu_{\text{island}}\partial^2/\partial x^2 & 0 \end{bmatrix}$, $\boldsymbol{G} = \begin{bmatrix} 0 & 0 \\ 0 & -1/D \end{bmatrix}$; $\boldsymbol{Z} = [w_{\text{island}}, \theta, T, M_y]^T$ is the state vector, in which $\theta = \partial w_{\text{island}}/\partial y$ and $T = -V_y$. The Hamiltonian operator matrix $\boldsymbol{H}$ satisfies $\boldsymbol{H}^T = \boldsymbol{JHJ}$, where $\boldsymbol{J} = \begin{bmatrix} 0 & \mathbf{I}_2 \\ -\mathbf{I}_2 & 0 \end{bmatrix}$ is the symplectic matrix[31] in which $\mathbf{I}_2$ is a 2 × 2 unit matrix.

3.2 Analytical solution approach

As shown in Fig. 5b–d, the superposition of two sub-problems (Fig. 5c and d) is proposed for the solution of the island's deformation (Fig. 5b) due to the symmetry of the structure as well as the boundary conditions and loads. One is the deformation of the island on the substrate slidingly supported along $x = 0$ and $x = a$ (*i.e.*, the slope $\partial w_{\text{island}}/\partial x$ and equivalent shear force V_x vanish along these edges), with a pair of opposite slopes denoted by φ_0 distributed along $y = 0$ and $y = b$ (Fig. 5c); the other one is the deformation of the same island slidingly supported along $y = 0$ and $y = b$, with φ_0 distributed along $x = 0$ and $x = a$ (Fig. 5d).

The solutions of the sub-problems start with solving the homogeneous eqn (14), which is explored by the method of separation of variables

$$\boldsymbol{Z} = \boldsymbol{X}(x)Y(y) \tag{15}$$

This gives

$$\frac{\mathrm{d}Y(y)}{\mathrm{d}y} = \mu Y(y), \quad \boldsymbol{HX}(x) = \mu \boldsymbol{X}(x) \qquad \text{(16a and b)}$$

where μ is the eigenvalue to be determined and $\boldsymbol{X}(x)$ is the eigenvector.

For the island with a pair of opposite edges (*e.g.*, $x = 0$ and $x = a$) slidingly supported, the eigenvalue problem (16b) gives two groups of eigenvalues (see Appendix for details)

$$\begin{aligned} &\mu_1 = \sqrt{R},\ \mu_2 = -\sqrt{R},\ \mu_3 = \sqrt{R}\mathrm{i},\ \mu_4 = -\sqrt{R}\mathrm{i} \\ &\mu_{n1} = \sqrt{{\alpha_n}^2 + R},\ \mu_{n2} = -\sqrt{{\alpha_n}^2 + R},\ \mu_{n3} = \sqrt{{\alpha_n}^2 - R}, \\ &\mu_{n4} = -\sqrt{{\alpha_n}^2 - R}\ \ (n = 1, 2, 3, \cdots) \end{aligned} \tag{17}$$

where $\alpha_n = n\pi/a$, $R = \sqrt{k/D}$ i, and i is the imaginary unit. The corresponding eigenvectors are

$$\begin{aligned} &\boldsymbol{X}_1(x) = \left[1,\ \sqrt{R},\ DR^{3/2},\ -DR\right]^T,\ \boldsymbol{X}_2(x) = \left[1,\ -\sqrt{R},\ -DR^{3/2},\ -DR\right]^T \\ &\boldsymbol{X}_3(x) = \left[1,\ \sqrt{R}\mathrm{i},\ -DR^{3/2}\mathrm{i},\ DR\right]^T,\ \boldsymbol{X}_4(x) = \left[1,\ -\sqrt{R}\mathrm{i},\ DR^{3/2}\mathrm{i},\ DR\right]^T \\ &\boldsymbol{X}_{\text{n}1}(x) = \{1,\ \sqrt{{\alpha_n}^2 + R},\ D[R + {\alpha_n}^2(\nu_{\text{island}} - 1)]\sqrt{{\alpha_n}^2 + R},\ D[R + {\alpha_n}^2(\nu_{\text{island}} - 1)]\}^T \cos(\alpha_n x) \\ &\boldsymbol{X}_{\text{n}2}(x) = \{1,\ -\sqrt{{\alpha_n}^2 + R},\ -D[R + {\alpha_n}^2(\nu_{\text{island}} - 1)]\sqrt{{\alpha_n}^2 + R},\ D[R + {\alpha_n}^2(\nu_{\text{island}} - 1)]\}^T \cos(\alpha_n x) \\ &\boldsymbol{X}_{\text{n}3}(x) = \{1,\ \sqrt{{\alpha_n}^2 - R},\ -D[R + {\alpha_n}^2(\nu_{\text{island}} - 1)]\sqrt{{\alpha_n}^2 - R},\ D[R + {\alpha_n}^2(\nu_{\text{island}} - 1)]\}^T \cos(\alpha_n x) \\ &\boldsymbol{X}_{\text{n}4}(x) = \{1,\ -\sqrt{{\alpha_n}^2 - R},\ D[R - {\alpha_n}^2(\nu_{\text{island}} - 1)]\sqrt{{\alpha_n}^2 - R},\ D[R + {\alpha_n}^2(\nu_{\text{island}} - 1)]\}^T \cos(\alpha_n x) \end{aligned} \tag{18}$$

The eigenvectors in eqn (18) satisfy the symplectic conjugacy and orthogonality, *i.e.*, $\int_0^a \boldsymbol{X}_1(x)^T\boldsymbol{J}\boldsymbol{X}_2(x)\mathrm{d}x \neq 0$, $\int_0^a \boldsymbol{X}_3(x)^T\boldsymbol{J}\boldsymbol{X}_4(x)\mathrm{d}x \neq 0$, $\int_0^a \boldsymbol{X}_{n1}(x)^T\boldsymbol{J}\boldsymbol{X}_{n2}(x)\mathrm{d}x \neq 0$, $\int_0^a \boldsymbol{X}_{n3}(x)^T\boldsymbol{J}\boldsymbol{X}_{n4}(x)\mathrm{d}x \neq 0$, and the other combinations of the eigenvectors are orthogonal.

The solution of eqn (14) is expanded according to the conjugacy and orthogonality by

$$\boldsymbol{Z} = \boldsymbol{X}(x)\boldsymbol{Y}(y) \tag{19}$$

where

$$\begin{aligned} \boldsymbol{X}(x) = [&\boldsymbol{X}_1(x), \boldsymbol{X}_2(x), \boldsymbol{X}_3(x), \boldsymbol{X}_4(x), \cdots,\ \boldsymbol{X}_{n1}(x), \boldsymbol{X}_{n2}(x), \boldsymbol{X}_{n3}(x), \\ &\boldsymbol{X}_{n4}(x), \cdots] \\ \boldsymbol{Y}(y) = [&Y_1(y), Y_2(y), Y_3(y), Y_4(y), \cdots,\ Y_{n1}(y), Y_{n2}(y), Y_{n3}(y), \\ &Y_{n4}(y), \cdots]^T \end{aligned} \tag{20}$$

From eqn (16a) and (17),

$$\begin{aligned} &Y_1 = c_1 \exp(\sqrt{R}y),\quad Y_2 = c_2 \exp(-\sqrt{R}y), \\ &Y_3 = c_3 \exp(\mathrm{i}\sqrt{R}y),\quad Y_4 = c_4 \exp(-\mathrm{i}\sqrt{R}y) \\ &Y_{n1} = c_{n1} \exp(\sqrt{{\alpha_n}^2 + R}y),\quad Y_{n2} = c_{n2} \exp(-\sqrt{{\alpha_n}^2 + R}y), \\ &Y_{n3} = c_{n3} \exp(\sqrt{{\alpha_n}^2 - R}y),\quad Y_{n4} = c_{n4} \exp(-\sqrt{{\alpha_n}^2 - R}y) \end{aligned} \tag{21}$$

where c_1–c_4 and c_{n1}–c_{n4} are the constants to be determined.

For the sub-problem 1 (Fig. 5c), after the Fourier series expansion of the slope φ_0, the boundary conditions at $y = 0$ and $y = b$ are

$$\begin{aligned} &V_y\big|_{y=0} = 0,\ \left.\frac{\partial w_{\text{island}}}{\partial y}\right|_{y=0} = \varphi_0 = \sum_{n=0,1,2,\cdots}^{\infty} E_n \cos(\alpha_n x), \\ &V_y\big|_{y=b} = 0,\ \left.\frac{\partial w_{\text{island}}}{\partial y}\right|_{y=b} = -\varphi_0 = -\sum_{n=0,1,2,\cdots}^{\infty} E_n \cos(\alpha_n x) \end{aligned} \tag{22}$$

Substituting eqn (18) and (21) into (20) then (19), the solution $\boldsymbol{Z}$ with c_1–c_4 and c_{n1}–c_{n4} is obtained, substituting which into eqn (22) yields those constants in terms of E_n (n = 0,1,2,…). It is noted that the cosine series expansion is imposed for φ_0 because comparison of the coefficients of $\cos(\alpha_n x)$ is necessary for obtaining E_n (n = 0,1,2,…). Accordingly, the solutions $\boldsymbol{Z}$ with E_n are obtained. The governing deflection solution, denoted by $w_1(x,y)$ and normalized by M_0a^2/D, is

$$w_1(x,y)=\frac{M_0a^2}{D}\times\left\{\begin{array}{l}\dfrac{\overline{E}_0}{2\left(-\dfrac{ka^4}{D}\right)^{\frac14}}\left\{\begin{array}{l}\csc\left(-\dfrac{ka^4}{D}\right)^{\frac14}\left\{\cos\left[\left(-\dfrac{ka^4}{D}\right)^{\frac14}\dfrac{y}{a}\right]+\cos\left[\left(-\dfrac{ka^4}{D}\right)^{\frac14}\left(1-\dfrac{y}{a}\right)\right]\right\}\\-\operatorname{csch}\left(-\dfrac{ka^4}{D}\right)^{\frac14}\left\{\cosh\left[\left(-\dfrac{ka^4}{D}\right)^{\frac14}\dfrac{y}{a}\right]+\cosh\left[\left(-\dfrac{ka^4}{D}\right)^{\frac14}\left(1-\dfrac{y}{a}\right)\right]\right\}\end{array}\right\}\\+\dfrac{1}{2\sqrt{\dfrac{ka^4}{D}}\mathrm{i}}\displaystyle\sum_{n=1}^{\infty}\overline{E}_n\left\{\begin{array}{l}\dfrac{\left[\sqrt{\dfrac{ka^4}{D}}\mathrm{i}-(n\pi)^2(1-\nu_{\mathrm{island}})\right]\left[e^{\sqrt{(n\pi)^2-\sqrt{\frac{ka^4}{D}}\mathrm{i}}\,\frac{y}{a}}+e^{\sqrt{(n\pi)^2-\sqrt{\frac{ka^4}{D}}\mathrm{i}}\left(1-\frac{y}{a}\right)}\right]}{\sqrt{(n\pi)^2-\sqrt{\dfrac{ka^4}{D}}\mathrm{i}}\left(1-e^{\sqrt{(n\pi)^2-\sqrt{\frac{ka^4}{D}}\mathrm{i}}}\right)}\\+\dfrac{\left[(n\pi)^2(1-\nu_{\mathrm{island}})+\sqrt{\dfrac{ka^4}{D}}\mathrm{i}\right]\left[e^{\sqrt{(n\pi)^2+\sqrt{\frac{ka^4}{D}}\mathrm{i}}\,\frac{y}{a}}+e^{\sqrt{(n\pi)^2+\sqrt{\frac{ka^4}{D}}\mathrm{i}}\left(1-\frac{y}{a}\right)}\right]}{\sqrt{(n\pi)^2+\sqrt{\dfrac{ka^4}{D}}\mathrm{i}}\left(1-e^{\sqrt{(n\pi)^2+\sqrt{\frac{ka^4}{D}}\mathrm{i}}}\right)}\end{array}\right\}\cos\left(n\pi\dfrac{x}{a}\right)\end{array}\right\} \tag{23}$$

where $\bar{E}_0 = DE_0/(aM_0)$ and $\bar{E}_n = DE_n/(aM_0)$ are the normalized constants to be determined. The governing deflection solution for the sub-problem 2 (Fig. 5d), denoted by $w_2(x,y)$, is obtained by the symmetry as

$$w_2(x,y) = w_1(y,x) \tag{24}$$

To satisfy the remaining boundary conditions at the four edges of the island, the sum of the bending moments for the two sub-problems at each edge must equal the corresponding moment imposed by the buckled bridge. Based on the symmetry, it just requires

$$\begin{aligned}\left(M_{y1}+M_{y2}\right)\big|_{y=0} &= M_0\left[H\left(x-\frac{a-b}{2}\right)-H\left(x-\frac{a+b}{2}\right)\right]\\ &= M_0\left\{\frac{b}{a}+4\times\sum_{n=1,2,3,\cdots}^{\infty}\left[\frac{1}{n\pi}\sin\left(\frac{n\pi b}{2a}\right)\cos\left(\frac{n\pi}{2}\right)\right]\cos\left(\frac{n\pi x}{a}\right)\right\}\end{aligned} \tag{25}$$

where H denotes the Heaviside theta function, and M_{y1} and M_{y2} are the bending moments with E_n for sub-problems 1 and 2, respectively, which can be directly acquired from the obtained solution $\boldsymbol{Z}$ or readily obtained by eqn (23) and (24) *via* eqn (13b). Eqn (25) gives

$$\frac{b}{a}+\frac{1}{2}\left\{4\nu_{\mathrm{island}}+\left(-\frac{ka^4}{D}\right)^{\frac14}\left\{\cot\left[\frac12\left(-\frac{ka^4}{D}\right)^{\frac14}\right]+\coth\left[\frac12\left(-\frac{ka^4}{D}\right)^{\frac14}\right]\right\}\right\}\overline{E}_0+\sum_{n=1}^{\infty}\left[\frac{2\nu_{\mathrm{island}}\dfrac{ka^4}{D}}{(n\pi)^2+\dfrac{ka^4}{D}}\right]\overline{E}_n=0 \tag{26a}$$

and

$$\begin{aligned}&\frac{1}{j\pi}\cos\frac{j\pi}{2}\sin\left(\frac{j\pi}{2}\frac{b}{a}\right)-\frac{\nu_{\mathrm{island}}\cos^2\left(\dfrac{j\pi}{2}\right)\dfrac{ka^4}{D}}{(j\pi)^4+\dfrac{ka^4}{D}}\overline{E}_0\\&+\frac{1}{8\sqrt{\dfrac{ka^4}{D}}\mathrm{i}}\left\{\begin{array}{l}\dfrac{\left[(1-\nu_{\mathrm{island}})(j\pi)^2-\sqrt{\dfrac{ka^4}{D}}\mathrm{i}\right]^2}{\sqrt{(j\pi)^2-\sqrt{\dfrac{ka^4}{D}}\mathrm{i}}}\coth\dfrac{\sqrt{(j\pi)^2-\sqrt{\dfrac{ka^4}{D}}\mathrm{i}}}{2}\\-\dfrac{\left[(1-\nu_{\mathrm{island}})(j\pi)^2+\sqrt{\dfrac{ka^4}{D}}\mathrm{i}\right]^2}{\sqrt{(j\pi)^2+\sqrt{\dfrac{ka^4}{D}}\mathrm{i}}}\coth\dfrac{\sqrt{(j\pi)^2+\sqrt{\dfrac{ka^4}{D}}\mathrm{i}}}{2}\end{array}\right\}\\&\times\overline{E}_j+\cos^2\frac{j\pi}{2}\sum_{n=1}^{\infty}\left[\frac{(1-\nu_{\mathrm{island}})^2j^2n^2\pi^4-\nu_{\mathrm{island}}\dfrac{ka^4}{D}}{(j^2+n^2)\pi^4+\dfrac{ka^4}{D}}\overline{E}_n\right]=0\\&\quad\text{for } j=1,2,3,\cdots\end{aligned} \tag{26b}$$

where $\alpha_j = j\pi/a$.

The linear algebraic eqn (26a) and (b) are solved to determine the normalized constants $\bar{E}_0$ and $\bar{E}_n$ ($n = 1,2,3,\ldots$), which depend only on three normalized parameters: ka^4/D, b/a and ν_{island}. Substituting the results into eqn (23) and (24), and the governing deflection solution for the island is obtained by

$$w_{\text{island}}(x,y) = w_1(x,y) + w_2(x,y) \tag{27}$$

which, normalized by M_0a^2/D, depends on the normalized coordinates x/a and y/a in addition to ka^4/D, b/a and ν_{island}. In calculation, a finite number of normalized constants $\bar{E}_0$, $\bar{E}_1$, $\bar{E}_2$, …, $\bar{E}_N$ (N is a positive integer) are obtained by taking the associated finite number of equations according to the convergence of the solution.

3.3 Scaling law for the maximum strain in the island

The maximum strain in the island is reached on the island surface at the bridge/island interconnection, and is given by[32]

$$\varepsilon_{\text{island}}^{\max} = -\frac{t_{\text{island}}}{2}\frac{\partial^2 w_{\text{island}}}{\partial x^2}\bigg|_{\left(0,\ \frac{a}{2}\right)} \tag{28}$$

Based on the results from eqn (27), (9), and (10), eqn (28) gives a scaling law for the maximum strain in the island,

$$\begin{aligned}\varepsilon_{\text{island}}^{\max} &= \frac{M_0 t_{\text{island}}}{D} g\left(\nu_{\text{island}}, \frac{b}{a}, \frac{ka^4}{D}\right)\\ &= \frac{E_{\text{bridge}} I_{\text{bridge}} t_{\text{island}}}{D L_{\text{bridge}}} f\left(\varepsilon_{\text{pre}}, \nu_{\text{island}}, \frac{b}{a}, \frac{ka^4}{D}\right)\end{aligned} \tag{29}$$

where g and f are non-dimensional functions.

Eqn (29) clearly shows that the maximum strain in the island, normalized by $E_{\text{bridge}}I_{\text{bridge}}t_{\text{island}}/(DL_{\text{bridge}})$, depends on the prestrain of the substrate *via* the island's Poisson ratio ν_{island}, normalized edge length b/a and normalized foundation modulus ka^4/D. This is shown in Fig. 6 for $\nu_{\text{island}} = 0.27$, $b/a = 0.2$ and $ka^4/D = 108\,445$, which correspond to $E_{\text{island}} = 130$ GPa, $a = 20$ μm, $b = 4$ μm and $t_{\text{island}} = 50$ nm for the silicon island[12,15] in stretchable electronics, and $k = E_{\text{substrate}}/\{2t_{\text{island}}(1-\nu_{\text{substrate}}{}^2)\{E_{\text{island}}(1-\nu_{\text{substrate}}{}^2)/[3E_{\text{substrate}}(1-\nu_{\text{island}}{}^2)]\}^{1/3}\} = 9.9 \times 10^{11}\ \text{N m}^{-3}$ (ref. 25–27) for the PDMS substrate with $\nu_{\text{substrate}} = 0.48$ and $E_{\text{substrate}} = 2$ MPa.[12,15] The convergent analytical results agree well with those obtained by FEM. A contour plot of the distribution of maximum strain $\varepsilon_{\text{island}}^{\max}$ in the island at $\varepsilon_{\text{pre}} = 25\%$ is included as an inset in Fig. 6.

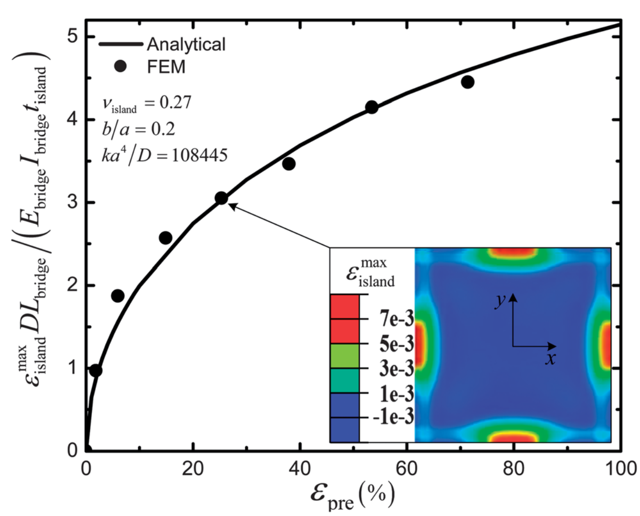

Fig. 6 Normalized maximum strain of the island *versus* the prestrain of the substrate.

4 Concluding remarks

The island-bridge structure plays an important role in stretchable electronics for its ability to achieve large and reversible stretchability. Strain control in the structure is necessary, especially for semiconductor device islands which normally withstand very small strain. The analytical mechanics model developed in this paper not only gives insight into the mechanical behavior of the structure but is also useful for the optimal design of the systems.

Deformation of the systems involves 10 quantities of the bridge, island and substrate: the lengths a and L_{bridge}, width b, thicknesses t_{bridge} and t_{island}, Young's moduli E_{bridge} and E_{island}, Poisson's ratio ν_{island}, foundation modulus k and prestrain ε_{pre}. The scaling law in eqn (29), verified by FEM, shows that the normalized maximum strain in the island depends only on the prestrain ε_{pre}, Poisson's ratio of island ν_{island}, normalized length b/a and normalized foundation modulus ka^4/D. This scaling law could serve as the theoretical basis for predicting and controlling the maximum strain in the islands.

Appendix: derivation of eqn (17)

The characteristic equation of the eigenvalue problem (16b) with the roots denoted by ξ gives $D(\xi^2 + \mu^2)^2 + k = 0$, which yields $\xi = \pm\sqrt{\mu^2 + R}\mathrm{i}$ and $\pm\sqrt{\mu^2 - R}\mathrm{i}$. Accordingly, the general solution for the eigenvector is

$$\begin{aligned}\boldsymbol{X}(x) = &\begin{pmatrix}A_1\\A_2\\A_3\\A_4\end{pmatrix}\cos\left(\sqrt{\mu^2+R}x\right) + \begin{pmatrix}B_1\\B_2\\B_3\\B_4\end{pmatrix}\sin\left(\sqrt{\mu^2+R}x\right)\\ &+ \begin{pmatrix}C_1\\C_2\\C_3\\C_4\end{pmatrix}\cos\left(\sqrt{\mu^2-R}x\right) + \begin{pmatrix}F_1\\F_2\\F_3\\F_4\end{pmatrix}\sin\left(\sqrt{\mu^2-R}x\right)\end{aligned} \tag{A1}$$

where A_n, B_n, C_n and F_n ($n = 1, 2, 3, 4, \ldots$) are constants.

Substituting eqn (A1) into eqn (16b) yields

$$\begin{aligned}&A_2 = \mu A_1,\ \ A_3 = D\mu A_1\left[R(\nu_{\text{island}} - 2) + \mu^2(\nu_{\text{island}} - 1)\right],\\ &A_4 = DA_1\left[\mu^2(\nu_{\text{island}} - 1) + R\nu_{\text{island}}\right],\ B_2 = \mu B_1,\\ &B_3 = D\mu B_1\left[R(\nu_{\text{island}} - 2) + \mu^2(\nu_{\text{island}} - 1)\right],\\ &B_4 = DB_1\left[\mu^2(\nu_{\text{island}} - 1) + R\nu_{\text{island}}\right],\ C_2 = \mu C_1,\\ &C_3 = D\mu C_1\left[R(2 - \nu_{\text{island}}) + \mu^2(\nu_{\text{island}} - 1)\right],\\ &C_4 = DC_1\left[\mu^2(\nu_{\text{island}} - 1) - R\nu_{\text{island}}\right],\ F_1 = \frac{F_2}{\mu},\\ &F_3 = DF_2\left[R(2 - \nu_{\text{island}}) + \mu^2(\nu_{\text{island}} - 1)\right],\\ &F_4 = \frac{DF_2}{\mu}\left[\mu^2(\nu_{\text{island}} - 1) - R\nu_{\text{island}}\right]\end{aligned} \tag{A2}$$

Impose the slidingly supported boundary conditions at a pair of opposite edges of the island, *e.g.*, $x = 0$ and $x = a$,

$$\left.\frac{\partial w_{\text{island}}}{\partial x}\right|_{x=0,a} = 0, \quad V_x|_{x=0,a} = 0 \tag{A3}$$

Substituting eqn (A1) and (A2) into eqn (A3), the existence of non-zero solutions for A_1, B_1, C_1 and F_2 requires $(\mu^4 - R^2)\sin(\sqrt{\mu^2 + R}a)\sin(\sqrt{\mu^2 - R}a) = 0$, which gives two groups of roots, *i.e.*, eigenvalues in eqn (17).

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References

1 T. Li, Z. Suo, S. P. Lacour and S. Wagner, *J. Mater. Res.*, 2005, **20**, 3274–3277.
2 A. Romeo, Q. Liu, Z. Suo and S. P. Lacour, *Appl. Phys. Lett.*, 2013, **102**, 131904.
3 D. Y. Khang, H. Jiang, Y. Huang and J. A. Rogers, *Science*, 2006, **311**, 208–212.
4 J. A. Rogers, T. Someya and Y. Huang, *Science*, 2010, **327**, 1603–1607.
5 T. Sekitani and T. Someya, *Adv. Mater.*, 2010, **22**, 2228–2246.
6 G. P. Crawford, *Flexible Flat Panel Display Technology*, Wiley, New York, 2005.
7 J. Yoon, A. J. Baca, S. I. Park, P. Elvikis, J. B. Geddes, L. F. Li, R. H. Kim, J. L. Xiao, S. D. Wang, T. H. Kim, M. J. Motala, B. Y. Ahn, E. B. Duoss, J. A. Lewis, R. G. Nuzzo, P. M. Ferreira, Y. G. Huang, A. Rockett and J. A. Rogers, *Nat. Mater.*, 2008, **7**, 907–915.
8 S. C. B. Mannsfeld, B. C. K. Tee, R. M. Stoltenberg, C. V. H. H. Chen, S. Barman, B. V. O. Muir, A. N. Sokolov, C. Reese and Z. N. Bao, *Nat. Mater.*, 2010, **9**, 859–864.
9 T. Sekitani, H. Nakajima, H. Maeda, T. Fukushima, T. Aida, K. Hata and T. Someya, *Nat. Mater.*, 2009, **8**, 494–499.
10 S.-W. Hwang, H. Tao, D.-H. Kim, H. Cheng, J.-K. Song, E. Rill, M. A. Brenckle, B. Panilaitis, S. M. Won, Y.-S. Kim, Y. M. Song, K. J. Yu, A. Ameen, R. Li, Y. Su, M. Yang, D. L. Kaplan, M. R. Zakin, M. J. Slepian, Y. Huang, F. G. Omenetto and J. A. Rogers, *Science*, 2012, **337**, 1640–1644.
11 R. Li, H. Cheng, Y. Su, S.-W. Hwang, L. Yin, H. Tao, M. A. Brenckle, D.-H. Kim, F. G. Omenetto, J. A. Rogers and Y. Huang, *Adv. Funct. Mater.*, 2013, **23**, 3106–3114.
12 H. C. Ko, M. P. Stoykovich, J. Song, V. Malyarchuk, W. M. Choi, C.-J. Yu, J. B. Geddes, J. Xiao, S. Wang, Y. Huang and J. A. Rogers, *Nature*, 2008, **454**, 748–753.
13 Y. M. Song, Y. Xie, V. Malyarchuk, J. Xiao, I. Jung, K.-J. Choi, Z. Liu, H. Park, C. Lu, R.-H. Kim, R. Li, K. B. Crozier, Y. Huang and J. A. Rogers, *Nature*, 2013, **497**, 95–99.
14 X. Feng, Y. Huang and A. J. Rosakis, *Int. J. Solids Struct.*, 2008, **45**, 3688–3698.
15 J. Song, Y. Huang, J. Xiao, S. Wang, K. C. Hwang, H. C. Ko, D.-H. Kim, M. P. Stoykovich and J. A. Rogers, *J. Appl. Phys.*, 2009, **105**, 123516.
16 Y. Huang, Z. Yin and Y. Xiong, *J. Appl. Mech.*, 2010, **77**, 041016.
17 B. Peng, Y. Huang, Z. Yin and Y. Xiong, *J. Appl. Phys.*, 2011, **110**, 073508.
18 R. Li, Y. Li, C. Lu, J. Song, R. Saeidpouraza, B. Fang, Y. Zhong, P. M. Ferreira, J. A. Rogers and Y. Huang, *Soft Matter*, 2012, **8**, 7122–7127.
19 D.-H. Kim, J. Song, W. M. Choi, H.-S. Kim, R.-H. Kim, Z. Liu, Y. Y. Huang, K.-C. Hwang, Y.-w. Zhang and J. A. Rogers, *Proc. Natl. Acad. Sci. U. S. A.*, 2008, **105**, 18675–18680.
20 Y. Huang, X. Wang, Y. Duan, N. Bu and Z. Yin, *Soft Matter*, 2012, **8**, 8302–8311.
21 S. P. Lacour, J. Jones, S. Wagner, T. Li and Z. Suo, *Proc. IEEE*, 2005, **93**, 1459–1467.
22 D.-H. Kim, J.-H. Ahn, W. M. Choi, H.-S. Kim, T.-H. Kim, J. Song, Y. Y. Huang, Z. Liu, C. Lu and J. A. Rogers, *Science*, 2008, **320**, 507–511.
23 Y. Su, J. Wu, Z. Fan, K.-C. Hwang, J. Song, Y. Huang and J. A. Rogers, *J. Mech. Phys. Solids*, 2012, **60**, 487–508.
24 A. Carlson, A. M. Bowen, Y. Huang, R. G. Nuzzo and J. A. Rogers, *Adv. Mater.*, 2012, **24**, 5284–5318.
25 E. Sultan and A. Boudaoud, *J. Appl. Mech.*, 2008, **75**, 051002.
26 J. Yin and X. Chen, *Philos. Mag. Lett.*, 2010, **90**, 423–433.
27 J. Y. Chung, A. J. Nolte and C. M. Stafford, *Adv. Mater.*, 2011, **23**, 349–368.
28 S. P. Timoshenko and J. M. Gere, *Theory of Elastic Stability*, McGraw-Hill, New York, 1961.
29 *ABAQUS Analysis User's Manual V6.9*, Dassault Systèmes, Pawtucket, 2009.
30 R. Li, Y. Zhong and M. Li, *Proc. R. Soc. A*, 2013, **469**, 20120681.
31 W. Yao, W. Zhong and C. W. Lim, *Symplectic Elasticity*, World Scientific, Singapore, 2009.
32 S. P. Timoshenko and S. W. Woinowsky-Krieger, *Theory of Plates and Shells*, McGraw-Hill, New York, 1959.

